



Binding Number Bounds of Zero Divisor Graphs of Classes of Completely Primary Finite Rings

Eliud Mmasi¹
Michael Onyango Ojiema²
Vincent Marani³

¹ *eliudmmasi@gmail.com*

² *mojiema@mmust.ac.ke*

³ *vmarani@kibu.ac.ke*

¹ *Department of Mathematics, Kibabii University, P. O. Box 1699 -50200, Bungoma, Kenya.*

² *Department of Mathematics, Masinde Muliro University of Science and Technology, P. O. Box 190-50100, Kakamega, Kenya.*

³ *Department of Mathematics, Kibabii University, P. O. Box 1699 -50200, Bungoma, Kenya.*

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ABSTRACT

The binding number of a graph is an important graph parameter which measures the distribution of the size of the graph and its related properties including toughness, rapture degree, scattering number and its integrity. For complete graphs $G \simeq K_n$ obtained from commutative finite ring, some results exist on the bounds of binding numbers. In this paper, we consider an incomplete but connected zero divisor graph $\Gamma(R)$ associated with a class of completely primary finite ring R and use standard procedures to compute the binding number bounds, the average binding number and related graph parameters.

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Keywords: *Binding number bounds, zero divisor graph, average binding number*



1 Introduction

The binding number of a graph is a concept that was first introduced by Woodall [18] in an attempt to measure distribution of the edges of a graph. In addition, various results have been obtained by a number of researchers on bounds, the binding numbers of specific graphs and the conditions on the binding number that guarantee the presence of a required subgraph. Goddard in [10] showed that for any graph G on m vertices, $\text{bind}(G \times K_n) = (nm - 1)/(nm - \delta(G) - n + 1)$ for all sufficiently large n and derived a general expression for the binding number of the lexicographic product $G[H]$ holding for all G and for H with $\text{bind}(H) \geq 1$.

The study of binding number in graphs is an important research area, perhaps also among the fastest growing area within graph theory. The reason for the steady and rapid growth of this area may be the diversity of its applications to both theoretical and real world problems. Recently, interest has been aroused that yielded a succession of results covering the cliques and cycles as well as the range values of binding numbers. Linzhung et al [12] extended the binding number to the edges and studied the edge-binding number of some plane graph. They found out that the average parameters to be more useful in some circumstances than the corresponding measures based on worst-case situations.

In graph theory, binding number refers to the minimum number of edges that need to be removed from a graph to make it disconnected. It represents the resilience or vulnerability of a graph, as a higher binding number indicates that more edges have to be removed to disconnect the graph. Bounds in the theory of graphs assist to establish limits or constraints on certain graph properties. Bounds are crucial in understanding and characterizing the behavior of graphs in different contexts. The study of the genus and bounds on a graph was done first by Tamizh in [17] when he analysed some geometric features of the total graph of an arbitrary commutative finite ring R and the set of $Z(R)$ as its set of all zero divisors.

Tamizh in [17] enumerated certain bounds which were both lower and upper for the genus of finite rings that are commutative for which total graph has two bounds. They defined a graph in which each pair of unique vertices is linked by an edge as a graph that is complete and denoted by K_n . A bipartite graph G is the one whose vector set $V(G)$ can be broken into two subsets V_1 and V_2 such that the set of edges of such graph consists those edges which precisely link the vertices in V_1 and V_2 . In particular if E consists of all possible such edges, then G is said to be a graph which is bipartite and complete denoted by the symbol $K_{m,n}$ where $|V_1| = m$ and $|V_2| = n$.

For an arbitrary commutative ring R with identity 1 as an ideal contained in $Z(R)$ and $|I| = \lambda \geq 3$ and $|R/I| = \mu$ then the lower bound for the total graph is given as follows, see [17].

$$(i) \text{ If } 2 \in I, \text{ then } g(T(R)) \geq \mu \left[\frac{(\lambda-3)(\lambda-4)}{12} \right].$$

$$(ii) \text{ If } 2 \notin I \text{ then } g(T(R)) \geq \left[\frac{(\lambda-3)(\lambda-4)}{12} \right] + \left(\frac{\mu-1}{2} \right) \left[\frac{(\lambda-2)^2}{4} \right].$$

The upper bound for the graph of the arbitrary ring was summarised in the following theorem :

Theorem 1.1. ,see [17], Suppose R is a commutative and $Z(R) = \bigcup_{i=1}^m t_i$ where t_i 's are ideals of R .

Assume that $|t_i| = \alpha_i$ and $|R_i|t_i| = \beta_i$ for $i = 1, \dots, m$

Suppose that $2 \in t_i$ for all $1 \leq t \leq j$ and $2 \notin t_i$ for all $j+1 \leq t \leq m$ where $j = 0$ if $2 \notin Z(R)$. Then

$$g(T(R)) \leq (m-1)(|R| - 1) + \sum_{t=1}^j \beta_t \left[\frac{(\alpha_t-3)(\alpha_t-4)}{12} \right] + \sum_{t=j+1}^m \left(\frac{\beta_t-1}{2} \right) \left[\frac{(\alpha_t-2)^2}{4} \right] + \left[\frac{(\alpha_t-3)(\alpha_t-4)}{12} \right].$$

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Some important references related to graphs over rings can be obtained from [1, 2, 5, 6, 7, 16]. Now, let G be a graph of order $V(G)$ and use $E(G)$. For $v \in V(G)$, the local binding number of v is

$$\text{bind}_v(G) = \min_{S \in F_v(G)} \left\{ \frac{|N(S)|}{|S|} \right\},$$

where $F_v(G) = \{S \subseteq V(G) \mid v \in S, S \neq \emptyset, N(S) \neq V(G)\}$. Clearly,

$$\text{bind}(G) = \min_{v \in V(G)} \{\text{bind}_v(G)\}.$$

A local binding set of v in G is $S \in F_v(G)$ such that $\text{bind}_v(G) = |N(S)|/|S|$. Furthermore, the average binding number of G is defined as

$$\text{bind}_{\text{av}}(G) = \frac{1}{n} \sum_{v \in V(G)} \text{bind}_v(G),$$

where n is the number of vertices in graph G [4].

In this study, binding number bounds of the zero divisor graphs $\Gamma(R)$ associated with finite rings are obtained. Furthermore, some results of the average binding number of the zero divisor graphs of classes of completely primary finite rings are generated. The related theorems of the average binding number and other graph parameters are provided.

2 Preliminaries

The following results are useful in the sequel.

Theorem 2.1. *If G is a graph of order n with the minimum degree $\delta(G)$, then*

$$\text{bind}_{\text{av}}(G) \geq \frac{\delta(G)}{n - \delta(G)}.$$

Proof. Let $v \in V(G)$ and S_v be a local binding set at v . Certainly $|N(S_v)| \geq d(v) \geq \delta(G)$. Since $N(S_v) \neq V(G)$, S_v omit all the neighbours of some vertex w , and $|S_v| \leq n - d(w) \leq n - \delta(G)$. Thus

$$\text{bind}_v(G) = \frac{|N(S_v)|}{|S_v|} \geq \frac{\delta(G)}{n - \delta(G)}.$$

Hence

$$\text{bind}_{\text{av}}(G) = \frac{1}{n} \sum_{v \in V(G)} \text{bind}_v(G) \geq \frac{\delta(G)}{n - \delta(G)}.$$

□

Theorem 2.2. *If G is a graph of order n with covering number $\alpha(G)$ and independence number $\beta(G)$, then*

$$\text{bind}_{\text{av}}(G) \leq \frac{\alpha(G)(\beta(G) + n - 1)}{n\beta(G)}.$$

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Proof. Let $v \in V(G)$ and M be a maximum independent set of G .

If $v \in V(M)$, then there is a local binding set $S_v \in F_v(G)$ that contains all vertices in the maximum independent set of G , and $|S_v| = \beta(G)$. Then $|N(S_v)| = \alpha(G)$ and $\text{bind}_v(G) = \alpha(G)/\beta(G)$.

If $v \notin V(M)$, then for $S \in F_v(G)$, $|S| \geq \beta(G)$ and $|N(S)| \leq n - 1$. Thus $\text{bind}_v(G) \leq (n - 1)/\beta(G)$. Hence

$$\begin{aligned} \text{bind}_{av}(G) &\leq \frac{\beta(G) \left(\frac{\alpha(G)}{\beta(G)} \right) + \alpha(G) \left(\frac{n-1}{\beta(G)} \right)}{n} \\ &= \frac{\alpha(G)(\beta(G) + n - 1)}{n\beta(G)}. \end{aligned}$$

□

Theorem 2.3. If G is a graph of order n with the minimum degree $\delta(G)$ and maximum degree $\Delta(G)$, then

$$\text{bind}_{av}(G) \geq \frac{\delta(G)}{\Delta(G) + 1}.$$

Proof. Let $v \in V(G)$. For $S_v \in F_v(G)$, $|S_v| \leq \Delta(G) + 1$ and $|N(S_v)| \geq \delta(G)$, hence

$$\text{bind}_v(G) = \min_{S_v \in F_v(G)} \frac{|N(S_v)|}{|S_v|} \geq \frac{\delta(G)}{\Delta(G) + 1}.$$

Thus

$$\text{bind}_{av}(G) = \frac{1}{n} \sum_{v \in V(G)} \text{bind}_v(G) \geq \frac{\delta(G)}{\Delta(G) + 1}.$$

□

Theorem 2.4. If H is a spanning subgraph of G , then

$$\text{bind}_{av}(H) \leq \text{bind}_{av}(G).$$

Proof. Let $v \in V(G) \cap V(H)$ with a local binding set $S_v^* \in F_v(G)$ of G . Let's denote $N_G(S_v^*) = N(S_v^*) \cap V(G)$ and $N_H(S_v^*) = N(S_v^*) \cap V(H)$. Then $N_H(S_v^*) \subseteq N_G(S_v^*)$ and

$$\text{bind}_v(H) \leq \frac{|N_H(S_v^*)|}{|S_v^*|} \leq \frac{|N_G(S_v^*)|}{|S_v^*|} = \text{bind}_v(G).$$

Thus by the definition, $\text{bind}_{av}(H) \leq \text{bind}_{av}(G)$.

□

Theorem 2.5. If G is a graph of order n , then

$$\text{bind}(G) \leq \text{bind}_{av}(G).$$

Proof. From the definitions it is clear that for $v \in V(G)$, $\text{bind}(G) \leq \text{bind}_v(G)$. Thus

$$\text{bind}(G) \leq \frac{1}{n} \sum_{v \in V(G)} \text{bind}_v(G) = \text{bind}_{av}(G).$$

□



Theorem 2.6. *If P_n is a path of order $n \geq 3$, then*

$$\text{bind}_{av}(P_n) = \begin{cases} 1, & n \text{ even,} \\ \frac{2n^2-5n+3}{2n^2-4n}, & n \text{ odd.} \end{cases}$$

Proof. Let $v \in V(P_n)$ and $S_v \in F_v(P_n)$. **Case 1:** n is even. Since $|N(S_v)| \geq |S_v|$ with the equality holds when S_v is a maximum independent set of P_n , therefore, $\text{bind}(P_n) = 1$ for all v , which implies $\text{bind}_{av}(P_n) = 1$.

Case 2: n is odd. Let the vertices of P_n be p_1, p_2, \dots, p_n in order along the path. There is a maximum independent set that gives the unique independence number $\beta(P_n) = \frac{1}{2}(n+1)$.

For any v in the maximum independent set containing $\frac{1}{2}(n+1)$ vertices, the $\text{bind}_v(P_n)$ is obtained when $|S_v| = \beta(P_n) = \frac{1}{2}(n+1)$ and $|N(S_v)| = \frac{1}{2}(n-1)$. Thus $\text{bind}_v(P_n) = (n-1)/(n+1)$.

For any v not in a maximum independent set of $\frac{1}{2}(n-1)$ vertices, to obtain $\text{bind}_v(P_n)$ we require S_v as large as possible without $N(S_v)$ being the whole of $V(P_n)$, i.e., $|S_v| = n-2$, for example, $S_v = \{p_3, p_4, \dots, p_n\}$. Thus $\text{bind}_v(P_n) = (n-1)/(n-2)$. Hence, by the definition

$$\begin{aligned} \text{bind}_{av}(P_n) &= \frac{1}{n} \left(\frac{(n+1)(n-1)}{2(n+1)} + \frac{(n-1)(n-1)}{2(n-2)} \right) \\ &= \frac{2n^2 - 5n + 3}{2n^2 - 4n} \end{aligned}$$

□

Theorem 2.7. *If C_n is a cycle of order $n \geq 4$, then*

$$\text{bind}_{av}(C_n) = \begin{cases} 1, & n \text{ even,} \\ \frac{n-1}{n-2}, & n \text{ odd.} \end{cases}$$

Proof. Let $v \in V(C_n)$ and $S_v \in F_v(C_n)$. **Case 1:** n is even. For all $v \in V(C_n)$, if $|S_v| = r$ then $|N(S_v)| \geq r$, therefore, $\text{bind}_v(C_n) \geq 1$. Since there is a local binding set S_v^* of C_n such that $|S_v^*| = \frac{1}{2}n$, when S_v^* is a maximum independent set of C_n , and $|N(S_v^*)| = \frac{1}{2}n$. Hence $\text{bind}_v(C_n) = 1$, for $v \in V(C_n)$, and therefore, $\text{bind}_{av}(C_n) = 1$

Case 2: n is odd. For $|S_v| = r$, then $r \leq n - \delta(C_n) = n-2$ and $|N(S_v)| \geq r+1$. Thus $|N(S_v)|/|S_v| \geq (r+1)/r$. is a decreasing function of r that has the minimum value when $r = n-2$. Hence $\text{bind}_v(C_n) \geq (n-1)/(n-2)$. Since there is a binding set S_v^* of C_n such that $|S_v^*| = n-2$ and $|N(S_v^*)| = n-1$, thus $\text{bind}(C_n) = (n-1)/(n-2)$, and therefore, $\text{bind}_{av}(C_n) = (n-1)/(n-2)$. □

Theorem 2.8. *If K_n is a complete graph of order $n \geq 2$, then $\text{bind}_{av}(K_n) = n-1$.*

Proof. Let $v \in V(K_n)$ and $S_v \in F_v(K_n)$. If $|S_v| \geq 2$, then $N(S_v) = V(K_n)$, which contradicts with $S_v \in F_v(K_n)$. Thus $|S_v| = 1$ and $|N(S_v)| = n-1$. Hence $\text{bind}_v(K_n) = n-1$ for all $v \in V(K_n)$ and $\text{bind}_{av}(K_n) = n-1$. □

Theorem 2.9. *If $K_{a,b}$ is a complete bipartite graph of order $a+b$ with $1 \leq a \leq b$, then*

$$\text{bind}_{av}(K_{a,b}) = 1.$$

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Proof. Let $V(K_{a,b}) = V(G_1) \cup V(G_2)$ be the vertex set of $K_{a,b}$, where the set $V(G_1)$ contains a vertices having degree b and the set $V(G_2)$ contains b vertices having degree a . For $v \in V(K_{a,b})$ and $S_v \in F_v(V(K_{a,b}))$, if $|S_v \cap V(G_1)| \neq 0$ and $|S_v \cap V(G_2)| \neq 0$ then $N(S_v) = V(K_{a,b})$, a contradiction. This implies that $S_v \subseteq V(G_1)$ to have $N(S_v) = V(G_2)$, or $S_v \subseteq V(G_2)$ to have $N(S_v) = V(G_1)$. Since for a vertices of $v \in V(G_1)$, $|S_v| \leq a$ and $|N(S_v)| = b$, i.e., $\text{bind}_v(K_{a,b}) = b/a$, and for b vertices of $v \in V(G_2)$, $|S_v| \leq b$ and $|N(S_v)| = a$, i.e., $\text{bind}_v(K_{a,b}) = a/b$. Therefore,

$$\text{bind}_{av}(K_{a,b}) = \frac{1}{a+b} \left(a \frac{b}{a} + b \frac{a}{b} \right) = 1.$$

□

3 A Survey on the Structures of the Zero Divisors of Rings of Idealization

3.1 Introduction

Suppose $GR(p^{sr}, p^s)$ is a ring which is Galois of characteristic p^s and order p^{sr} ie $GR(p^{sr}, p^s) = \mathbb{Z}_{p^s}[x]/(f(x))$, where $f \in \mathbb{Z}_{p^s}[x]$ is a monic polynomial of degree r whose image in $\mathbb{Z}_{p^s}[x]$ is irreducible. Suppose $\text{char } R = p^s$ then R has a coefficient subring R_0 of the form $GR(p^{sr}, p^s)$ which is clearly a Galois subring of R which is maximal. Moreover there exists elements $m_1, m_2, \dots, m_n \in Z(R)$ and automorphism $\delta_1, \delta_2, \dots, \delta_n \in \text{Aut}(R_0)$ such that $R = R_0 \oplus \sum_{i=1}^h R_0 M_i$ as $(R_0-$ modules), $m_i r = r^{\delta} m_i$ for every $r \in R_0$ and any $i = 1, \dots, h$. Furthermore $\delta_1, \dots, \delta_h$ are uniquely determined by R and R_0 . The ideal of R which is maximal is $Z(R) = pR_0 \oplus \sum_{i=1}^h R_0 M_i$. From the construction above it is seen that $R = R_0 \oplus U \oplus V \oplus W$ and $R^* = \mathbb{Z}_{p^{r-1}} \times (1 + Z(R))$ which is a cartesian product of the Abelian groups $Z(R) = pR_0 \oplus U \oplus V \oplus W$ where the R_0- modules U, V and W are finitely generated. It is well known that Galois Rings are the trivial finite rings which are completely Primary in whose class lie the square radical zero finite rings [13], the cube radical zero finite rings [9], the power four radical zero finite rings [14], and the five radical index finite rings [11] which are rings of idealization.

3.2 Square Radical Zero Finite Rings

In [13], the authors constructed commutative completely primary finite $R = R_0 \oplus U$ where U is a h -dimensional R_0 -module with a unique maximal ideal $Z(R)$ such that $Z(R) \neq (0)$ and $(Z(R))^2 = (0)$. It was established that the Ring constructed was finite of characteristic p or p^2 with

$$Z(R) = pR_0 \oplus U$$

and

$$(Z(R))^2 = (0).$$



3.3 Cube Radical Zero Finite Rings

Chikunji in [9] constructed the ring $R = R_0 \oplus U \oplus V$ in whose subset of all the zero divisors forms a unique maximal ideal and $(Z(R))^3 = 0$ and $(Z(R))^2 \neq 0$. The ring constructed is completely primary finite of characteristics p, p^2 or p^3 whose structure of the zero divisors is as follows:

$$(i) Z(R) = pR_0 \oplus \sum_{i=1}^s R_0 u_i \oplus \sum_{i,j=1}^s R_0 u_i u_j$$

$$(ii) Z(R)^2 = p^2 R_0 \oplus \sum_{i,j=1}^s R_0 u_i u_j$$

$$\text{or } (Z(R))^2 = pR_0 \oplus \sum_{i,j=1}^s R_0 u_i u_j$$

$$(iii) (Z(R))^3 = (0).$$

3.4 Four Index Radical Commutative Completely Primary Finite Rings

Owino and Ojiema in [14] constructed rings $R = R' \oplus U \oplus V \oplus W$ in which the product of any four Zero divisors is Zero. Moreover their group of units have been characterized. From the well known Raghavendran results [15], we see that $(Z(R))^4 = (0)$, $(Z(R))^3 \neq (0)$ then the characteristic of R is p, p^2, p^3 and p^4 . Rings with the above characteristics were classified as follows:

3.4.1 Rings of characteristic p

Let $R' = GR(p^r, p)$ be the Galois ring of order p^r and characteristic p . Suppose U, V and W are finitely generated R' modules such that $\dim_{R'}(U) = s, \dim_{R'}(V) = t$ and $\dim_{R'}(W) = \lambda$ and $s + t + \lambda = h$.

Suppose $\{u_1, \dots, u_s\}, \{v_1, \dots, v_t\}$ and $\{w_1, \dots, w_\lambda\}$ are the generators of $U, V,$ and W respectively so that $R = R' \oplus U \oplus V \oplus W$ is an additive abelian group, see [14].

Further assume that $s = 1, t = 1, \lambda = h - 2$ so that $R = R' \oplus R' u \oplus R' v \oplus \sum_{j=1}^{h-2} R' w_j$ and $pu = pv = pw_j = 0$ $1 \leq j \leq h - 2$.

On R define multiplication as $(r_0, r_1, \dots, r_h)((s_0, s_1, \dots, s_h) = (r_0 s_0, r_0 s_1 + r_1 s_0, r_0 s_2 + r_2 s_0 + r_1 s_1, r_0 s_3 + r_3 s_0 + r_1 s_2 + r_2 s_1, \dots, r_0 s_h + r_h s_0 + r_1 s_2 + r_2 s_1)$.

It was easy to verify that the given multiplication turned R to be a commutative ring with unit element as $(1, 0, \dots, 0)$.

The structures of the zero divisors are as follows:

$$(i) Z(R) = R' u \oplus R' v \oplus \sum_{j=1}^{\lambda} R' w_j.$$

$$(ii) (Z(R))^2 = R' v \oplus \sum_{j=1}^{\lambda} R' w_j.$$

$$(iii) (Z(R))^3 = \sum_{j=1}^{\lambda} R' w_j.$$

$$iv (Z(R))^4 = (0).$$



3.4.2 Rings of characteristic p^2

Suppose $R' = GR(p^{2r}, p^2)$ is a Galois ring of order p^{2r} and characteristic p^2 . Suppose U, V and W are finitely generated R' modules such that $\dim_{R'}(U) = s, \dim_{R'}(V) = t$ and $\dim_{R'}(W) = \lambda$ and $s + t + \lambda = h$ as seen in [14]

Let $\{u_1, \dots, u_s\}, \{v_1, \dots, v_t\}$ and $\{w_1, \dots, w_\lambda\}$ be the generators of $U, V,$ and W respectively so that $R = R' \oplus U \oplus V \oplus W$ is an additive Abelian group.

Further assume that $t = h - 1, t = 1 \lambda = 0$ so that $R = R' \oplus \sum_{j=1}^{h-1} R' u_j \oplus R' u \oplus R' v$ where $pu_j \neq 0, t^2 u_j = 0, 1 \leq j \leq s$ and $tv = 0$.

On R let multiplication be defined as $(r_0, r_1, \dots, r_{h-1}, \bar{r}_h)((k_0, k_1, \dots, k_{h-1}, \bar{k}_h) = (r_0 s_0 + t \sum_{i,j=1}^{h-1} r_i s_j, r_0 s_1 + r_1 s_0, \dots, r_0 s_{h-1} + r_{h-1} s_0, r_0 \bar{s}_h + \bar{r}_h s_0)$ where $\bar{r}_h, \bar{k}_h \in R'/tR'$.

It is easy to confirm that the given multiplication turns R into a commutative ring with unit element as $(1, 0, \dots, \bar{0})$.

The structure of the zero divisors is as follows:

- (i) $Z(R) = pR' \oplus \sum_{j=1}^s R' u_j \oplus R' v.$
- (ii) $(Z(R))^2 = pR' \oplus t \sum_{j=1}^s R' u_j \oplus R' v.$
- (iii) $(Z(R))^3 = t \sum_{j=1}^s R' u_j.$
- iv $(Z(R))^4 = (0).$

3.4.3 Rings of characteristic p^3

Let $R' = GR(p^{3r}, p^3)$ be the Galois ring of order p^{3r} and characteristic p^3 . Suppose U, V and W are finitely generated R' modules such that $\dim_{R'}(U) = s, \dim_{R'}(V) = t$ and $\dim_{R'}(W) = \lambda$ and $s + t + \lambda = h$ and $s + t + \lambda = h$.

Let $\{u_1, \dots, u_s\}, \{v_1, \dots, v_t\}$ and $\{w_1, \dots, w_\lambda\}$ be the generators of $U, V,$ and W respectively so that $R = R' \oplus U \oplus V \oplus W$ is an additive abelian group, see [14].

Further assume that $s = h - 1, t = 1 \lambda = 0$ so that $R = R' \oplus \sum_{j=1}^{h-1} R' u_j \oplus R' v$ where $tu_j \neq 0, t^3 u_j = 0, 1 \leq j \leq s$ and $tv = 0$.

On R define multiplication as $(r_0, \bar{r}_1, \bar{r}_2, \dots, \bar{r}_{h-1}, \bar{r}_h)((s_0, \bar{s}_1, \dots, \bar{s}_{h-1}, \bar{s}_h) = (r_0 s_0, r_0 \bar{s}_1 + \bar{r}_1 s_0, \dots, r_0 \bar{s}_{h-1} + \bar{r}_{h-1} s_0, \dots, r_0 \bar{s}_h + \bar{r}_h s_0 + \sum_{i,j=1}^{h-1} \bar{r}_i \bar{s}_j)$. Where $\bar{r}_h, \bar{s}_h \in R'/p^2 R'$ and $\bar{r}_h, \bar{s}_h \in R'/pR', 1 \leq j \leq s$

It can be shown that the the given multiplication converts R into a commutative ring with unit element as $(1, \bar{0}, \dots, \bar{0})$.

The structure of the zero divisors is as follows:

- (i) $Z(R) = pR' \oplus \sum_{j=1}^s R' u_j \oplus R' v.$
- (ii) $(Z(R))^2 = p^2 R' \oplus t \sum_{j=1}^s R' u_j \oplus R' v.$
- (iii) $(Z(R))^3 = tR' v.$
- iv $(Z(R))^4 = (0).$



3.4.4 Rings of characteristic p^4

Let $R' = GR(p^{4r}, p^4)$ be the Galois ring of order p^{4r} and characteristic p^4 . Suppose U, V and W are finitely generated R' modules such that $\dim_{R'}(U) = k, \dim_{R'}(V) = p$ and $\dim_{R'}(W) = \lambda$ and $s + t + \lambda = h$.

Let $\{u_1, \dots, u_s\}, \{v_1, \dots, v_t\}$ and $\{w_1, \dots, w_\lambda\}$ be the generators of $U, V,$ and W respectively so that $R = R' \oplus U \oplus V \oplus W$ is an additive abelian group, see [14].

Further assume that $k = h, t = 0, \lambda = 0$ so that $R = R' \oplus \sum_{j=1}^s R' u_j$ where $pu_j = 0, 1 \leq j \leq s$.

On R define multiplication as $(r_0, \bar{r}_1, \bar{r}_2, \dots, \bar{r}_h)((k_0, \bar{k}_1, \dots, \bar{k}_h) = (r_0 s_0, r_0 \bar{k}_1 + \bar{r}_1 k_0, \dots, r_0 \bar{k}_h + \bar{r}_h k_0)$. Where $\bar{r}_i, \bar{k}_j \in R'/tR', 1 \leq i, j \leq s$.

It can be shown that the given multiplication converts R into a ring which is commutative with unit element $(1, \bar{0}, \dots, \bar{0}, \bar{0})$.

The structure of the zero divisors is as follows:

$$(i) Z(R) = pR' \oplus \sum_{j=1}^s R' u_j.$$

$$(ii) (Z(R))^2 = p^2 R'.$$

$$(iii) (Z(R))^3 = p^3 R'.$$

$$iv) (Z(R))^4 = (0).$$

3.5 Five Index Radical Commutative Completely Primary Finite Rings

These classes of finite rings were recently constructed in [11] and characterised for the unit groups. The construction in [11] followed the following fashion:

3.5.1 Construction

For $1 \leq s \leq 5, t, r \in \mathbb{Z}^+$, let $R_0 = GR(t^{sr}, t^s)$ be a Galois ring. Suppose U, V, W and Y are R/pR_0 -spaces considered as R_0 -modules generated by e, f, g and h elements respectively such that the corresponding generating sets are $\{u_i, \dots, u_e\}, \{v_j, \dots, v_f\}, \{w_k, \dots, w_g\}$ and y_l, \dots, y_h so that $R = R_0 \oplus U \oplus V \oplus W \oplus Y$ is an additive abelian group. Then on the additive group, multiplication is defined in [11] as follows:

$$\begin{aligned} & (r_0 + \sum_{i=1}^e r_i u_i + \sum_{j=1}^f s_j v_j + \sum_{k=1}^g t_k w_k + \sum_{l=1}^h z_l y_l)(r'_0 + \sum_{i=1}^e r'_i u_i + \sum_{j=1}^f s'_j v_j + \sum_{k=1}^g t'_k w_k + \sum_{l=1}^h z'_l y_l) \\ &= r_0 r'_0 + p^a \sum_{i,m=1}^e (r_i r'_m + pR_0) + \sum_{i=1}^e [r_0 r'_i + r_i r'_0 + pR_0] u_i + \sum_{j=1}^f [(r_0 + pR_0) s'_j + s_j (r'_0 + pR_0) + \\ & \sum_{\nu, \mu=1}^e (r_\nu r'_\mu + pR_0)] v_j + \sum_{k=1}^g [(r_0 + pR_0) t'_k + t_k (r'_0 + pR_0) + \sum_{i,j} (r_i + pR_0) s'_j + s_j (r'_i + pR_0)] w_k + \end{aligned}$$

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$$\sum_{l=1}^h [(r_o + pR_0)z'_l + z_l(r'_o + pR_0)] + \sum_{i,k} (r_i + pR_0)t'_k + t_k(r'_i + pR_0) + \sum_{\kappa,\tau=1}^f (s_\kappa s'_\tau + pR_0)y_l$$

where $a = 1, 2, 3$ or 4 depending on whether $\text{char}R = p^2, p^3, p^4$ or p^5 . This multiplication turns R into a commutative ring with identity $(1, 0, 0, 0, 0)$.

If $R_0 = GR(p^r, p)$ where $\text{Char}R = p$, then the multiplication reduces to

$$\begin{aligned} & (r_o + \sum_{i=1}^e r_i u_i + \sum_{j=1}^f s_j v_j + \sum_{k=1}^g t_k w_k + \sum_{l=1}^h z_l y_l)(r'_o + \sum_{i=1}^e r'_i u_i + \sum_{j=1}^f s'_j v_j + \sum_{k=1}^g t'_k w_k + \sum_{l=1}^h z'_l y_l) \\ &= r_o r'_o + \sum_{i=1}^e [r_o r'_i + r_i r'_o] u_i + \sum_{j=1}^f [(r_o s'_j + s_j r'_o) + \sum_{\alpha,\mu=1}^e (r_\alpha r'_\mu)] v_j + \sum_{k=1}^g [(r_o t'_k + t_k r'_o) + \\ & \sum_{i,j=1}^e (r_i s'_j + s_j r'_i)] w_k + \sum_{l=1}^h [(r_o z'_l + z_l r'_o) + \sum_{i,k=1}^e (r_i t'_k + t_k r'_i) + \sum_{\kappa,\tau=1}^f (s_\kappa s'_\tau)] y_l \end{aligned}$$

4 Binding Numbers of the Zero Divisor Graphs

4.1 Rings of characteristic p

The structures of zero divisors of this class of rings is given by:

- (i) $Z(R) = U \oplus V \oplus W \oplus Y$
- (ii) $(Z(R))^2 = V \oplus W \oplus Y$
- (iii) $(Z(R))^3 = W \oplus Y$
- (iv) $(Z(R))^4 = Y$
- (v) $(Z(R))^5 = (0)$

Given a zero divisor graph $\Gamma(R)$ with vertex set $V(\Gamma(R))$. If x and y are any two vertices of the graph, then x and y lie in the edge of the graph $E(\Gamma(R))$ if and only if $xy = 0$. Using this adjacency property, we have the following results:

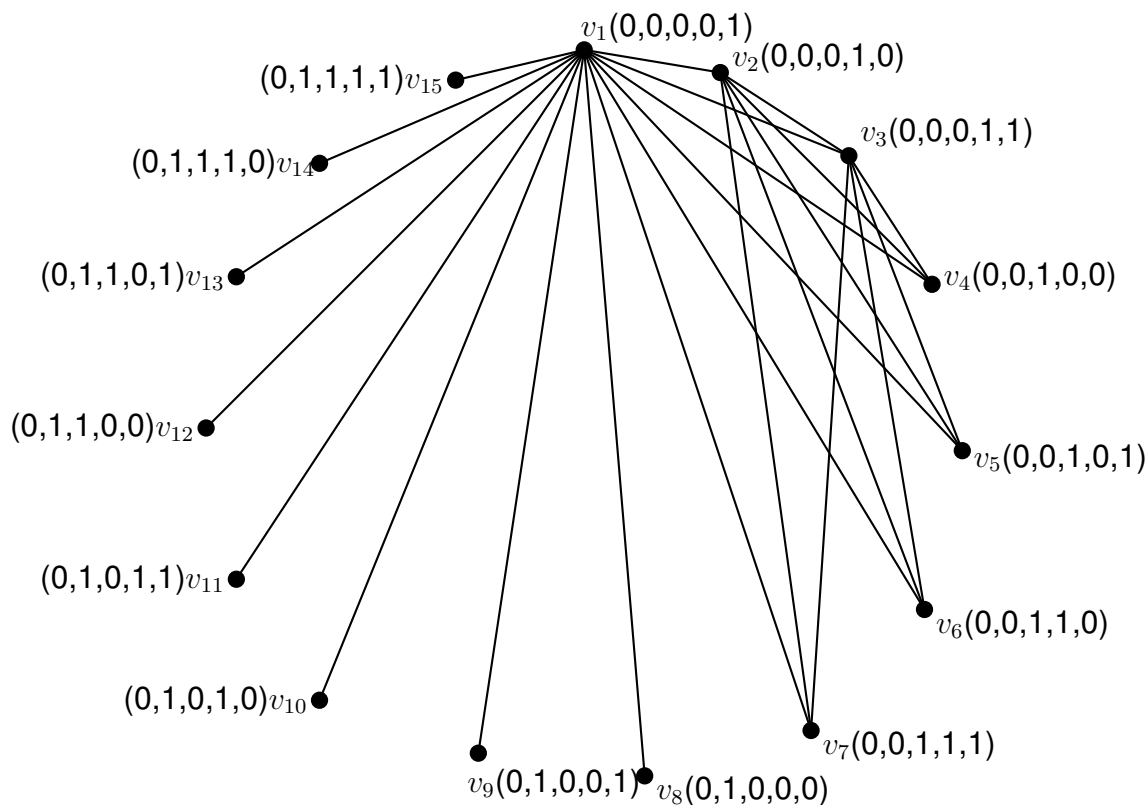
Example 4.1. If R is the ring of the construction above, where $R = R_0 \oplus U \oplus V \oplus W \oplus Y$ then for $\text{char}R = p = 2$ $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ then the zero divisors will be

$Z(R) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ In this case $p = 2, r = 1, e = 1, f = 1, g = 1$ and $h = 1$.

So the $\Gamma(R)$ is 4-partite with $\dim(\Gamma(R)) = 2, gr(\Gamma(R)) = 3$ and $b(\Gamma(R)) = \frac{1}{4}$.



When $\text{char} R = p = 2$ the graph is as follows;



Proposition 4.1. Let R be the ring with $pu_i = 0, pv_j = 0, pw_k = 0, py_l = 0, 1 \leq i \leq e, 1 \leq j \leq f, 1 \leq k \leq g$ and $1 \leq l \leq h$. Then the graph of the ring $R, \Gamma(R)$ satisfies the following :

- (i) $|V(\Gamma(R))| = p^{(e+f+g+h)r} - 1$
- (ii) bindingnumber, $b(\Gamma(R)) = \frac{p^{(g+h)r} - 1}{p^{(g+h)r}(p^{(e+f)r} - 1)}$
- (iii) $\Gamma(R)$ is $p^{(g+h)r}$ partite

Proof. Since $Z(R) = U \oplus V \oplus W \oplus Y$, we have $|Z(R)| = p^{(e+f+g+h)r}$ and because the set $Z(R)^* = Z(R) - \{(0, 0, 0, 0, 0)\}$, it follows that $|(Z(R))^*| = |V(\Gamma(R))| = p^{(e+f+g+h)r} - 1$ which establishes (i).

Let $\epsilon_1, \dots, \epsilon_r \in R_0$ with $\epsilon_1 = 1$ such that $\bar{\epsilon}_1, \dots, \bar{\epsilon}_r \in R_0/pR_0$ form a basis for R_0/pR_0 regarded as a vector space over its prime subfield \mathbb{F}_p . Also let

$$H_{\mu,k,l} = \left\{ \sum_{k=1}^g \sum_{\mu=1}^r \epsilon_{\mu} w_k + \sum_{l=1}^h \sum_{\mu=1}^r \epsilon_{\mu} y_l \right\}$$



Then the set $Z(R)^*$ is partitioned into the following mutually disjoint subsets;

$$\begin{aligned} & \{V_{\sum_{k=1}^g \sum_{\mu=1}^r \epsilon_{\mu} w_k + \sum_{l=1}^h \sum_{\mu=1}^r \epsilon_{\mu} y_l}\} = H_{\mu,k,l} - \{(0, 0, 0, 0, 0)\} \\ V_1 = & \{V_{\sum_{i=1}^e \sum_{\mu=1}^r \epsilon_{\mu} u_i + \sum_{j=1}^f \sum_{\mu=1}^r \epsilon_{\mu} v_j + \sum_{k=1}^g \sum_{\mu=1}^r \epsilon_{\mu} w_k + \sum_{l=1}^h \sum_{\mu=1}^r \epsilon_{\mu} y_l}\} = Z(R^*) - \\ & \bigcup_{\mu,k,j} \{V_{\sum_{k=1}^g \sum_{\mu=1}^r \epsilon_{\mu} w_k + \sum_{l=1}^h \sum_{\mu=1}^r \epsilon_{\mu} y_l}\} \end{aligned}$$

From the definition of V_1 , we have the set of neighbours of V_1 as

$$N(V_1) = \bigcup_{\mu,k,j} \{V_{\sum_{k=1}^g \sum_{\mu=1}^r \epsilon_{\mu} w_k + \sum_{l=1}^h \sum_{\mu=1}^r \epsilon_{\mu} y_l}\}$$

Clearly $|N(V_1)| = |\bigcup_{\mu,k,j} \{V_{\sum_{k=1}^g \sum_{\mu=1}^r \epsilon_{\mu} w_k + \sum_{l=1}^h \sum_{\mu=1}^r \epsilon_{\mu} y_l}\}| = p^{(g+h)r} - 1$. Therefore

$$\begin{aligned} |\widetilde{V_1}| &= |Z(R)| - |N(V_1)| = p^{(e+f+g+h)r} - 1 - (p^{(g+h)r} - 1) \\ &= p^{(g+h)r} (p^{(e+f)r} - 1) \end{aligned}$$

Thus the binding number of the graph $\Gamma(R)$ is given by

$$b(\Gamma(R)) = \frac{|N(V_1)|}{V_1} = \frac{p^{(g+h)r} - 1}{p^{(g+h)r} (p^{(e+f)r} - 1)}$$

Since $N(V_1) \neq \Phi$ and $V_1 \cap N(V_1) = \Phi$ then the sets V_1 and $N(V_1)$ are mutually disjoint and as a result $Z(R^*) = V_1 \cup N(V_1)$. In addition $|\bigcup(N(V_1))| = p^{(g+h)r} - 1$ and hence $\Gamma(R)$ is $p^{(g+h)r}$ partite. \square

4.2 Rings of characteristic p^2

The structures of zero divisors of this class of rings is given by:

- (i) $Z(R) = pR_0 \oplus U \oplus V \oplus W \oplus Y$
- (ii) $(Z(R))^2 = pR_0 \oplus U \oplus V \oplus W \oplus Y$
- (iii) $(Z(R))^3 = pU \oplus pV \oplus W \oplus Y$
- (iv) $(Z(R))^4 = pV \oplus Y$
- (v) $(Z(R))^5 = (0)$

Given a zero divisor graph $\Gamma(R)$ with vertex set $V(\Gamma(R))$. If x and y are any two vertices of the graph, then x and y lie in the edge of the graph $E(\Gamma(R))$ if and only if $xy = 0$. Using this adjacency property, we have the following results:

Example 4.2. Let $Z(R) = 2Z(R) = \mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus (Z)_2$. In this case $p = 2, r = 1, e = 1, f = 1, g = 1, g = 1$ and $h = 1$. Then the set of vertices $V(\Gamma(R))$ is given by

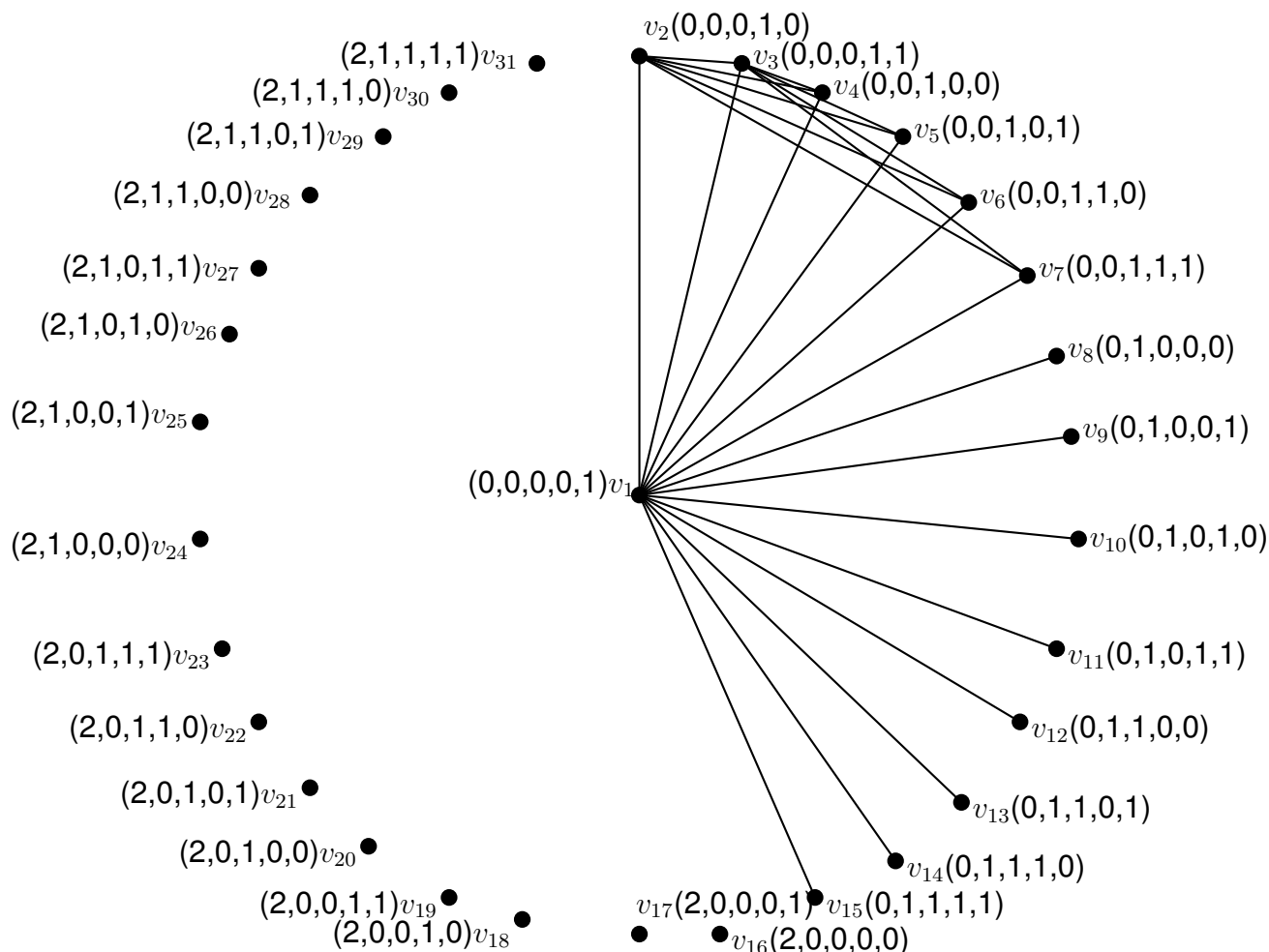
Let $Z(R) = 2Z(R) = \mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus (Z)_2$. In this case $p = 2, r = 1, e = 1, f = 1, g = 1, g = 1$ and $h = 1$. Then

$\Gamma(R)$ is 8-partite with $diam(\Gamma(R)) = 2, gr(\Gamma(R)) = 3$ and $b(\Gamma(R)) = \frac{7}{24}$.

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When $char R = P^2$ the graph is as follows;



Proposition 4.2. Let R be the ring constructed in [11] with $p^\zeta u_i = 0 \zeta = 1, 2; pv_j = 0, pw_k = 0, py_l = 0, 1 \leq i \leq e, 1 \leq j \leq f, 1 \leq k \leq g$ and $1 \leq l \leq h$. Then the graph of the ring $R, \Gamma(R)$ satisfies the following:

(i) $|V(\Gamma(R))| = p^{(1+\zeta e+f+g+h)r} - 1$

(ii) $b(\Gamma(R)) =$

$$\begin{cases} \frac{p^{(l+g+h)r} - 1}{p^{(l+g+h)r} (p^{(e+f)r} - 1)} & \text{if } \zeta = 1 \\ \frac{p^{(l+e+g+h)r} - 1}{(p^{(l+e+g+h)r} (p^{(e+f)r} - 1))} & \text{if } \zeta = 2. \end{cases}$$

(iii)

$$\Gamma(R) = \begin{cases} p^{(l+g+h)r} - \text{partite}, & \text{if } \zeta = 1 \\ p^{(l+e+g+h)r} - \text{partite}, & \text{if } \zeta = 2. \end{cases}$$

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Proof. Since $Z(R) = pR_0 \oplus U \oplus V \oplus W \oplus Y$, we have $|Z(R)| = p^{(1+\zeta e+f+g+h)r}$ and because the set $Z(R)^* = Z(R) - \{(0, 0, 0, 0, 0)\}$, it follows that $|(Z(R))^*| = |V(\Gamma(R))| = p^{(1+\zeta e+f+g+h)r} - 1$ which establishes (i).

To establish (ii) and (iii), let $\epsilon_1, \dots, \epsilon_r \in R_0$ with $\epsilon_1 = 1$ such that $\bar{\epsilon}_1, \dots, \bar{\epsilon}_r \in R_0/pR_0$ form a basis for R_0/pR_0 regarded as a vector space over its prime subfield \mathbb{F}_p . For $\zeta = 1$, let

$$H_{\mu,k,l} = \left\{ \sum_{\mu=1}^r m_{\mu} \epsilon_{\mu} + \sum_{k=1}^g \sum_{\mu=1}^r \epsilon_{\mu} w_k + \sum_{l=1}^h \sum_{\mu=1}^r \epsilon_{\mu} y_l, m_{\mu} \in \{0, \lambda(p)\} : 1 \leq \lambda \leq p-1 \right\}$$

Then the set $Z(R)^*$ is partitioned into the following mutually disjoint subsets;

$$\left\{ V_{\sum_{\mu=1}^r m_{\mu} \epsilon_{\mu} + \sum_{k=1}^g \sum_{\mu=1}^r \epsilon_{\mu} w_k + \sum_{l=1}^h \sum_{\mu=1}^r \epsilon_{\mu} y_l} \right\} = H_{\mu,k,l} - \{(0, 0, 0, 0, 0)\}$$

$$V_1 = \left\{ V_{\sum_{\mu=1}^r m_{\mu} \epsilon_{\mu} + \sum_{i=1}^e \sum_{\mu=1}^r \epsilon_{\mu} u_i + \sum_{j=1}^f \sum_{\mu=1}^r \epsilon_{\mu} v_j + \sum_{k=1}^g \sum_{\mu=1}^r \epsilon_{\mu} w_k + \sum_{l=1}^h \sum_{\mu=1}^r \epsilon_{\mu} y_l} \right\} = Z(R^*) - \bigcup_{\mu,k,j} \left\{ V_{\sum_{\mu=1}^r m_{\mu} \epsilon_{\mu} + \sum_{k=1}^g \sum_{\mu=1}^r \epsilon_{\mu} w_k + \sum_{l=1}^h \sum_{\mu=1}^r \epsilon_{\mu} y_l} \right\}$$

From the definition of V_1 , we have the set of neighbours of V_1 as

$$N(V_1) = \bigcup_{\mu,k,j} \left\{ V_{\sum_{\mu=1}^r m_{\mu} \epsilon_{\mu} + \sum_{k=1}^g \sum_{\mu=1}^r \epsilon_{\mu} w_k + \sum_{l=1}^h \sum_{\mu=1}^r \epsilon_{\mu} y_l} \right\}$$

Clearly $|N(V_1)| = \left| \bigcup_{\mu,k,j} \left\{ V_{\sum_{\mu=1}^r m_{\mu} \epsilon_{\mu} + \sum_{k=1}^g \sum_{\mu=1}^r \epsilon_{\mu} w_k + \sum_{l=1}^h \sum_{\mu=1}^r \epsilon_{\mu} y_l} \right\} \right| = p^{(l+g+h)r} - 1$. Therefore

$$\begin{aligned} |V_1| &= |Z(R)| - |N(V_1)| = p^{(l+e+f+g+h)r} - 1 - (p^{l+g+h}r - 1) \\ &= p^{(l+g+h)r} (p^{(e+f)r} - 1) \end{aligned}$$

Thus the binding number of the graph $\Gamma(R)$ is given by

$$b(\Gamma(R)) = \frac{|N(V_1)|}{|V_1|} = \frac{p^{(l+g+h)r} - 1}{p^{(l+g+h)r} (p^{(e+f)r} - 1)}$$

Since $N(V_1) \neq \Phi$ and $V_1 \cap N(V_1) = \Phi$ then the sets V_1 and $N(V_1)$ are mutually disjoint and as a result $Z(R^*) = V_1 \cup N(V_1)$. In addition $|\bigcup(N(V_1))| = p^{(l+g+h)r} - 1$ and hence $\Gamma(R)$ is $p^{(l+g+h)r}$ partite.

For $\zeta = 2$ let

$$H_{\mu,k,l} = \left\{ \sum_{\mu=1}^r m_{\mu} \epsilon_{\mu} + p \sum_{i=1}^e \sum_{\mu=1}^r \epsilon_{\mu} u_i + \sum_{k=1}^g \sum_{\mu=1}^r \epsilon_{\mu} w_k + \sum_{l=1}^h \sum_{\mu=1}^r \epsilon_{\mu} y_l, m_{\mu} \in \{0, \lambda(p)\} : 1 \leq \lambda \leq p-1 \right\}$$

Then the set $Z(R)^*$ is partitioned into the following mutually disjoint subsets;

$$\left\{ V_{\sum_{\mu=1}^r m_{\mu} \epsilon_{\mu} + p \sum_{i=1}^e \sum_{\mu=1}^r \epsilon_{\mu} u_i + \sum_{k=1}^g \sum_{\mu=1}^r \epsilon_{\mu} w_k + \sum_{l=1}^h \sum_{\mu=1}^r \epsilon_{\mu} y_l} \right\} = H_{\mu,k,l} - \{(0, 0, 0, 0, 0)\}$$



$$V_1 = \left\{ V_{\sum_{\mu=1}^r m_{\mu} \epsilon_{\mu} + \sum_{i=1}^e \sum_{\mu=1}^r \epsilon_{\mu} u_i + \sum_{j=1}^f \sum_{\mu=1}^r \epsilon_{\mu} v_j + \sum_{k=1}^g \sum_{\mu=1}^r \epsilon_{\mu} w_k + \sum_{l=1}^h \sum_{\mu=1}^r \epsilon_{\mu} y_l} \right\} = Z(R^*) - \bigcup_{\mu, k, j} \left\{ V_{\sum_{\mu=1}^r m_{\mu} \epsilon_{\mu} + p \sum_{i=1}^e \sum_{\mu=1}^r \epsilon_{\mu} u_i + \sum_{k=1}^g \sum_{\mu=1}^r \epsilon_{\mu} w_k + \sum_{l=1}^h \sum_{\mu=1}^r \epsilon_{\mu} y_l} \right\}$$

Clearly $|N(V_1)| = \left| \bigcup_{\mu, k, j} \left\{ V_{\sum_{\mu=1}^r m_{\mu} \epsilon_{\mu} + p \sum_{i=1}^e \sum_{\mu=1}^r \epsilon_{\mu} u_i + \sum_{k=1}^g \sum_{\mu=1}^r \epsilon_{\mu} w_k + \sum_{l=1}^h \sum_{\mu=1}^r \epsilon_{\mu} y_l} \right\} \right| = p^{(l+e+g+h)r} - 1$.
Therefore

$$\begin{aligned} |V_1| &= |Z(R)^*| - |N(V_1)| = p^{(l+2e+f+g+h)r} - 1 - (p^{l+e+g+h} - 1) \\ &= p^{(l+e+g+h)r} (p^{(e+f)r} - 1) \end{aligned}$$

Thus the binding number of the graph $\Gamma(R)$ is given by

$$b(\Gamma(R)) = \frac{|N(V_1)|}{|V_1|} = \frac{p^{(l+e+g+h)r} - 1}{p^{(l+e+g+h)r} (p^{(e+f)r} - 1)}$$

Since $N(V_1) \neq \Phi$ and $V_1 \cap N(V_1) = \Phi$ then the sets V_1 and $N(V_1)$ are mutually disjoint and as a result $Z(R^*) = V_1 \cup N(V_1)$. In addition $|\bigcup(N(V_1))| = p^{(l+e+g+h)r} - 1$ and hence $\Gamma(R)$ is $p^{(l+e+g+h)r}$ partite. \square

4.3 Rings of characteristic p^3

The structures of zero divisors of this class of rings is given by:

- (i) $Z(R) = pR_0 \oplus U \oplus V \oplus W \oplus Y$
- (ii) $(Z(R))^2 = pR_0 \oplus pU \oplus V \oplus W \oplus Y$
- (iii) $(Z(R))^3 = p^2U \oplus pV \oplus W \oplus Y$
- (iv) $(Z(R))^4 = p^2V \oplus pW \oplus Y$
- (v) $(Z(R))^5 = (0)$

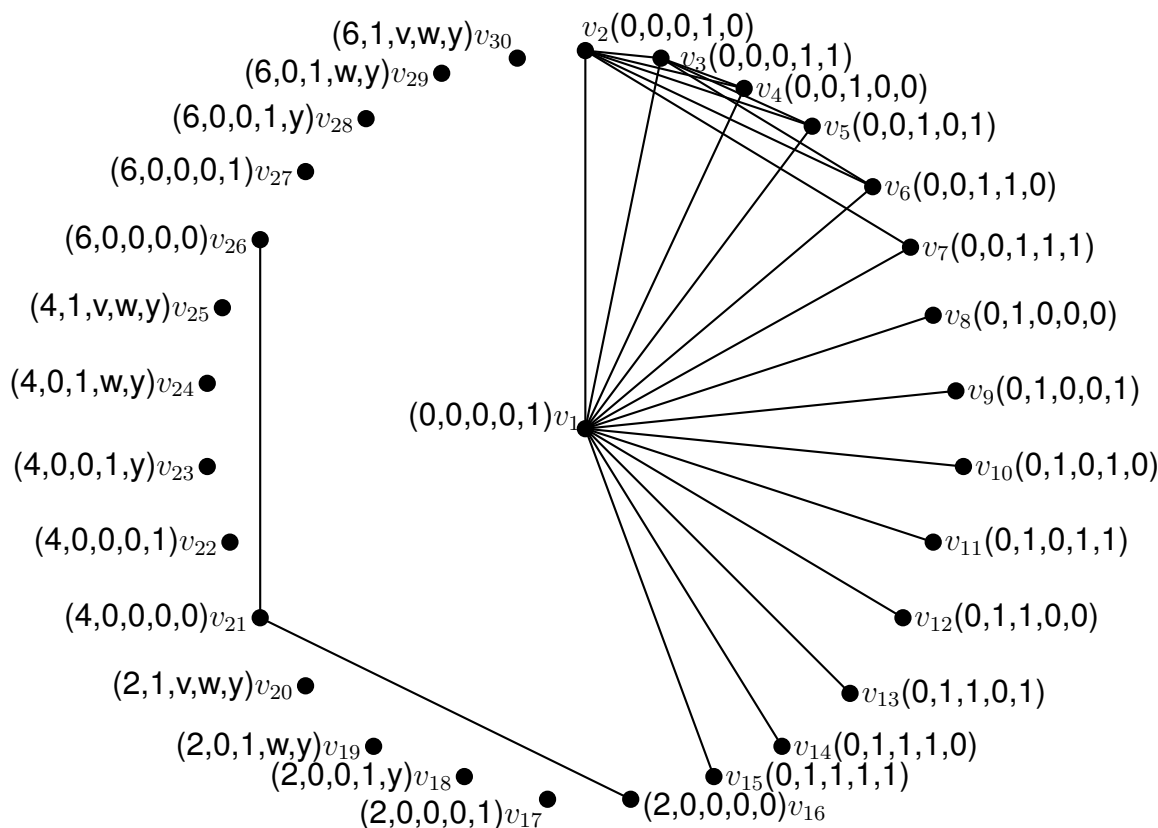
Given a zero divisor graph $\Gamma(R)$ with vertex set $V(\Gamma(R))$. If x and y are any two vertices of the graph, then x and y lie in the edge of the graph $E(\Gamma(R))$ if and only if $xy = 0$. Using this adjacency property, we have the following results:

Example 4.3. Let $Z(R) = 2Z(R) = 2\mathbb{Z}_8 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus (Z)_2$. In this case $p = 2, r = 1, e = 1, f = 1, g = 1, h = 1$ and $h = 1$. Then

$\Gamma(R)$ is 16-partite with $diam(\Gamma(R)) = 2, gr(\Gamma(R)) = 3$ and $b(\Gamma(R)) = \frac{5}{16}$.



When $char R = p^3$ the graph will appear as below;



Proposition 4.3. Let R be the ring constructed in [11] with $p^\zeta u_i = 0 \ \zeta = 1, 2, 3; pv_j = pw_k = py_l = 0, 1 \leq i \leq e, 1 \leq j \leq f, 1 \leq k \leq g$ and $1 \leq l \leq h$. Then the graph of the ring $R, \Gamma(R)$ satisfies the following:

(i) $|V(\Gamma(R))| = p^{(2+\zeta e+f+g+h)r} - 1$

(ii) $b(\Gamma(R)) =$

$$\begin{cases} \frac{p^{(2+g+h)r}-1}{p^{(2+g+h)r}(p^{(e+f)r}-1)} & \text{if; } \zeta = 1 \\ \frac{p^{(2+e+g+h)r}-1}{(p^{(2+e+g+h)r}(p^{(e+f)r}-1))} & \text{if; } \zeta = 2 \\ \frac{p^{(2+2e+g+h)r}-1}{(p^{(2+2e+g+h)r}(p^{(e+f)r}-1))} & \text{if; } \zeta = 3. \end{cases}$$

(iii)

$$\Gamma(R) = \begin{cases} p^{(2+g+h)r} - \text{partite,} & \text{if; } \zeta = 1 \\ p^{(2+e+g+h)r} - \text{partite,} & \text{if; } \zeta = 2 \\ p^{(2+2e+g+h)r} - \text{partite,} & \text{if; } \zeta = 3. \end{cases}$$

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Proof. Since $Z(R) = pR_0 \oplus U \oplus V \oplus W \oplus Y$, we have $|Z(R)| = p^{(2+\zeta e+f+g+h)r}$ and because the set $Z(R)^* = Z(R) - \{(0, 0, 0, 0, 0)\}$, it follows that $|(Z(R))^*| = |V(\Gamma(R))| = p^{(2+\zeta e+f+g+h)r} - 1$ which establishes (i).

To establish (ii) and (iii), let $\epsilon_1, \dots, \epsilon_r \in R_0$ with $\epsilon_1 = 1$ such that $\bar{\epsilon}_1, \dots, \bar{\epsilon}_r \in R_0/pR_0$ form a basis for R_0/pR_0 regarded as a vector space over its prime subfield \mathbb{F}_p . For $\zeta = 1$, let

$$H_{\mu,k,l} = \left\{ \sum_{\mu=1}^r m_{\mu} \epsilon_{\mu} + \sum_{k=1}^g \sum_{\mu=1}^r \epsilon_{\mu} w_k + \sum_{l=1}^h \sum_{\mu=1}^r \epsilon_{\mu} y_l, m_{\mu} \in \{0, (\lambda-1)(p)\} : 2 \leq \lambda \leq p^2 \right\}$$

Then the set $Z(R)^*$ is partitioned into the following mutually disjoint subsets;

$$\left\{ V_{\sum_{\mu=1}^r m_{\mu} \epsilon_{\mu} + \sum_{k=1}^g \sum_{\mu=1}^r \epsilon_{\mu} w_k + \sum_{l=1}^h \sum_{\mu=1}^r \epsilon_{\mu} y_l} \right\} = H_{\mu,k,l} - \{(0, 0, 0, 0, 0)\}$$

$$V_1 = \left\{ V_{\sum_{\mu=1}^r m_{\mu} \epsilon_{\mu} + \sum_{i=1}^e \sum_{\mu=1}^r \epsilon_{\mu} u_i + \sum_{j=1}^f \sum_{\mu=1}^r \epsilon_{\mu} v_j + \sum_{k=1}^g \sum_{\mu=1}^r \epsilon_{\mu} w_k + \sum_{l=1}^h \sum_{\mu=1}^r \epsilon_{\mu} y_l} \right\} = Z(R^*) - \bigcup_{\mu,k,j} \left\{ V_{\sum_{\mu=1}^r m_{\mu} \epsilon_{\mu} + \sum_{k=1}^g \sum_{\mu=1}^r \epsilon_{\mu} w_k + \sum_{l=1}^h \sum_{\mu=1}^r \epsilon_{\mu} y_l} \right\}$$

From the definition of V_1 , we have the set of neighbours of V_1 as

$$N(V_1) = \bigcup_{\mu,k,j} \left\{ V_{\sum_{\mu=1}^r m_{\mu} \epsilon_{\mu} + \sum_{k=1}^g \sum_{\mu=1}^r \epsilon_{\mu} w_k + \sum_{l=1}^h \sum_{\mu=1}^r \epsilon_{\mu} y_l} \right\}$$

Clearly $|N(V_1)| = \left| \bigcup_{\mu,k,j} \left\{ V_{\sum_{\mu=1}^r m_{\mu} \epsilon_{\mu} + \sum_{k=1}^g \sum_{\mu=1}^r \epsilon_{\mu} w_k + \sum_{l=1}^h \sum_{\mu=1}^r \epsilon_{\mu} y_l} \right\} \right| = p^{(2+g+h)r} - 1$. Therefore

$$\begin{aligned} |V_1| &= |Z(R)^*| - |N(V_1)| = p^{(2+e+f+g+h)r} - 1 - (p^{(2+g+h)r} - 1) \\ &= p^{(2+g+h)r} (p^{(e+f)r} - 1) \end{aligned}$$

Thus the binding number of the graph $\Gamma(R)$ is given by

$$b(\Gamma(R)) = \frac{|N(V_1)|}{V_1} = \frac{p^{(2+g+h)r} - 1}{p^{(2+g+h)r} (p^{(e+f)r} - 1)}$$

Since $N(V_1) \neq \Phi$ and $V_1 \cap N(V_1) = \Phi$ then the sets V_1 and $N(V_1)$ are mutually disjoint and as a result $Z(R^*) = V_1 \cup N(V_1)$. In addition $|\bigcup(N(V_1))| = p^{(2+g+h)r} - 1$ and hence $\Gamma(R)$ is $p^{(2+g+h)r}$ partite.

For $\zeta = 2$ let

$$H_{\mu,k,l} = \left\{ \sum_{\mu=1}^r m_{\mu} \epsilon_{\mu} + p \sum_{i=1}^e \sum_{\mu=1}^r \epsilon_{\mu} u_i + \sum_{k=1}^g \sum_{\mu=1}^r \epsilon_{\mu} w_k + \sum_{l=1}^h \sum_{\mu=1}^r \epsilon_{\mu} y_l, m_{\mu} \in \{0, (\lambda-)(p)\} : 2 \leq \lambda \leq p^2 \right\}$$

Then the set $Z(R)^*$ is partitioned into the following mutually disjoint subsets;

$$\left\{ V_{\sum_{\mu=1}^r m_{\mu} \epsilon_{\mu} + p \sum_{i=1}^e \sum_{\mu=1}^r \epsilon_{\mu} u_i + \sum_{k=1}^g \sum_{\mu=1}^r \epsilon_{\mu} w_k + \sum_{l=1}^h \sum_{\mu=1}^r \epsilon_{\mu} y_l} \right\} = H_{\mu,k,l} - \{(0, 0, 0, 0, 0)\}$$



$$V_1 = \left\{ V_{\sum_{\mu=1}^r m_{\mu} \epsilon_{\mu} + \sum_{i=1}^e \sum_{\mu=1}^r \epsilon_{\mu} u_i + \sum_{j=1}^f \sum_{\mu=1}^r \epsilon_{\mu} v_j + \sum_{k=1}^g \sum_{\mu=1}^r \epsilon_{\mu} w_k + \sum_{l=1}^h \sum_{\mu=1}^r \epsilon_{\mu} y_l} \right\} = Z(R^*) - \bigcup_{\mu, k, j} \left\{ V_{\sum_{\mu=1}^r m_{\mu} \epsilon_{\mu} + p \sum_{i=1}^e \sum_{\mu=1}^r \epsilon_{\mu} u_i + \sum_{k=1}^g \sum_{\mu=1}^r \epsilon_{\mu} w_k + \sum_{l=1}^h \sum_{\mu=1}^r \epsilon_{\mu} y_l} \right\}$$

Clearly $|N(V_1)| = \left| \bigcup_{\mu, k, j} \left\{ V_{\sum_{\mu=1}^r m_{\mu} \epsilon_{\mu} + p \sum_{i=1}^e \sum_{\mu=1}^r \epsilon_{\mu} u_i + \sum_{k=1}^g \sum_{\mu=1}^r \epsilon_{\mu} w_k + \sum_{l=1}^h \sum_{\mu=1}^r \epsilon_{\mu} y_l} \right\} \right| = p^{(2+e+g+h)r} - 1$.
Therefore

$$\begin{aligned} |V_1| &= |Z(R)^*| - |N(V_1)| = p^{(2+2e+f+g+h)r} - 1 - (p^{2+e+g+h}r - 1) \\ &= p^{(2+e+g+h)r} (p^{(e+f)r} - 1) \end{aligned}$$

Thus the binding number of the graph $\Gamma(R)$ is given by

$$b(\Gamma(R)) = \frac{|N(V_1)|}{V_1} = \frac{p^{(2+e+g+h)r} - 1}{p^{(2+e+g+h)r} (p^{(e+f)r} - 1)}$$

Since $N(V_1) \neq \Phi$ and $V_1 \cap N(V_1) = \Phi$ then the sets V_1 and $N(V_1)$ are mutually disjoint and as a result $Z(R^*) = V_1 \cup N(V_1)$. In addition $|\bigcup(N(V_1))| = p^{(2+e+g+h)r} - 1$ and hence $\Gamma(R)$ is $p^{(2+e+g+h)r}$ partite.

For $\zeta = 3$ $Z(R)^*$ is partitioned as in the case when $\zeta = 2$ but with

$$|N(V_1)| = \left| \bigcup_{\mu, k, j} \left\{ V_{\sum_{\mu=1}^r m_{\mu} \epsilon_{\mu} + p \sum_{i=1}^e \sum_{\mu=1}^r \epsilon_{\mu} u_i + \sum_{k=1}^g \sum_{\mu=1}^r \epsilon_{\mu} w_k + \sum_{l=1}^h \sum_{\mu=1}^r \epsilon_{\mu} y_l} \right\} \right| = p^{(2+2e+g+h)r} - 1 \text{ and}$$

$$\begin{aligned} |V_1| &= |Z(R)^*| - |N(V_1)| = p^{(2+3e+f+g+h)r} - 1 - (p^{2+2e+g+h}r - 1) \\ &= p^{(2+2e+g+h)r} (p^{(e+f)r} - 1) \end{aligned}$$

Thus the binding number of the graph $\Gamma(R)$ is given by

$$b(\Gamma(R)) = \frac{|N(V_1)|}{V_1} = \frac{p^{(2+2e+g+h)r} - 1}{p^{(2+2e+g+h)r} (p^{(e+f)r} - 1)}$$

Since $N(V_1) \neq \Phi$ and $V_1 \cap N(V_1) = \Phi$ then the sets V_1 and $N(V_1)$ are mutually disjoint and as a result $Z(R^*) = V_1 \cup N(V_1)$. In addition $|\bigcup(N(V_1))| = p^{(2+2e+g+h)r} - 1$ and hence $\Gamma(R)$ is $p^{(2+2e+g+h)r}$ partite. \square

4.4 Rings of characteristic p^4

The structures of zero divisors of this class of rings is given by:

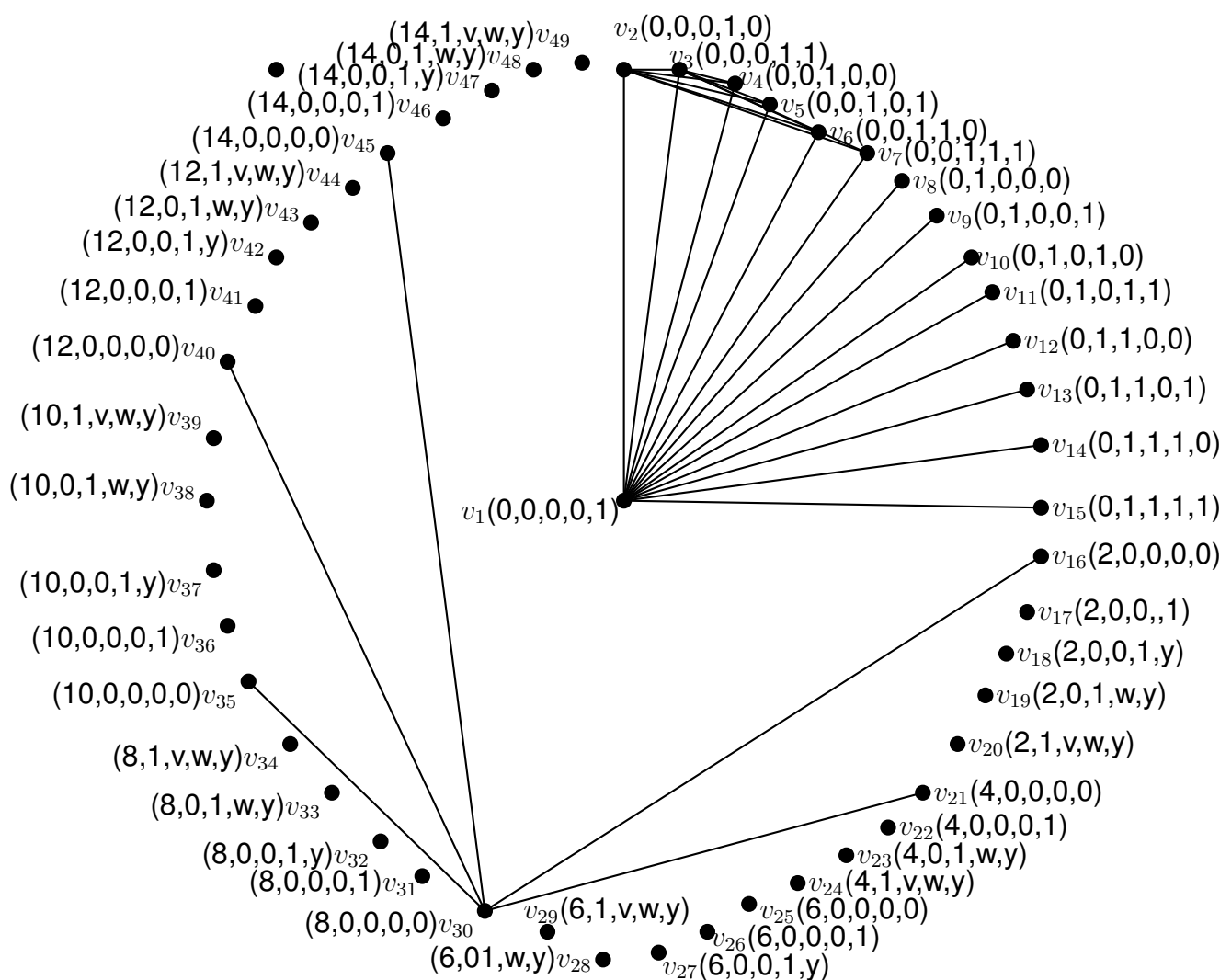
- (i) $Z(R) = pR_0 \oplus U \oplus V \oplus W \oplus Y$
- (ii) $(Z(R))^2 = pR_0 \oplus pU \oplus V \oplus W \oplus Y$
- (iii) $(Z(R))^3 = p^2R_0 \oplus p^2U \oplus pV \oplus W \oplus Y$
- (iv) $(Z(R))^4 = p^3U \oplus p^2V \oplus pW \oplus Y$
- (v) $(Z(R))^5 = (0)$



Given a zero divisor graph $\Gamma(R)$ with vertex set $V(\Gamma(R))$. If x and y are any two vertices of the graph, then x and y lie in the edge of the graph $E(\Gamma(R))$ if and only if $xy = 0$. Using this adjacency property, we have the following results:

Example 4.4. Let $Z(R) = 2\mathbb{Z}_{16} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus (Z)_2$. In this case $p = 2, r = 1, e = 1, f = 1, g = 1, g = 1$ and $h = 1$. Then

$\Gamma(R)$ is 32-partite with $diam(\Gamma(R)) = 2, gr(\Gamma(R)) = 3$ and $b(\Gamma(R)) = \frac{31}{96}$.
When $char R = p^4$ the graph is as follows;





Proposition 4.4. Let R be the ring constructed in [11] with $p^\zeta u_i = 0$ $\zeta = 1, 2, 3, 4$; $pv_j = pw_k = py_l = 0$, $1 \leq i \leq e$, $1 \leq j \leq f$, $1 \leq k \leq g$ and $1 \leq l \leq h$. Then the graph of the ring R , $\Gamma(R)$ satisfies the following:

(i) $|V(\Gamma(R))| = p^{(3+\zeta e+f+g+h)r} - 1$

(ii) $b(\Gamma(R)) =$

$$\begin{cases} \frac{p^{(3+g+h)r} - 1}{p^{(3+g+h)r} (p^{(e+f)r} - 1)} & \text{if; } \zeta = 1 \\ \frac{p^{(3+e+g+h)r} - 1}{(p^{(3+e+g+h)r} (p^{(e+f)r} - 1))} & \text{if; } \zeta = 2 \\ \frac{p^{(3+2e+g+h)r} - 1}{(p^{(3+2e+g+h)r} (p^{(e+f)r} - 1))} & \text{if; } \zeta = 3 \\ \frac{p^{(3+3e+g+h)r} - 1}{(p^{(3+3e+g+h)r} (p^{(e+f)r} - 1))} & \text{if; } \zeta = 4. \end{cases}$$

(iii)

$$\Gamma(R) = \begin{cases} p^{(3+g+h)r} - \text{partite}, & \text{if; } \zeta = 1 \\ p^{(3+e+g+h)r} - \text{partite}, & \text{if; } \zeta = 2 \\ p^{(3+2e+g+h)r} - \text{partite}, & \text{if; } \zeta = 3 \\ p^{(3+3e+g+h)r} - \text{partite}, & \text{if; } \zeta = 4. \end{cases}$$

Proof. Since $Z(R) = pR_0 \oplus U \oplus V \oplus W \oplus Y$, we have $|Z(R)| = p^{(3+\zeta e+f+g+h)r}$ and because the set $Z(R)^* = Z(R) - \{(0, 0, 0, 0, 0)\}$, it follows that $|(Z(R))^*| = |V(\Gamma(R))| = p^{(3+\zeta e+f+g+h)r} - 1$ which establishes (i).

To establish (ii) and (iii), let $\epsilon_1, \dots, \epsilon_r \in R_0$ with $\epsilon_1 = 1$ such that $\bar{\epsilon}_1, \dots, \bar{\epsilon}_r \in R_0/pR_0$ form a basis for R_0/pR_0 regarded as a vector space over its prime subfield \mathbb{F}_p . For $\zeta = 1$, let

$$H_{\mu,k,l} = \left\{ \sum_{\mu=1}^r m_\mu \epsilon_\mu + \sum_{k=1}^g \sum_{\mu=1}^r \epsilon_\mu w_k + \sum_{l=1}^h \sum_{\mu=1}^r \epsilon_\mu y_l, m_\mu \in \{0, \lambda(p)\} : 1 \leq \lambda \leq p^3 - 1 \right\}$$

Then the set $Z(R)^*$ is partitioned into the following mutually disjoint subsets;

$$\left\{ V_{\sum_{\mu=1}^r m_\mu \epsilon_\mu + \sum_{k=1}^g \sum_{\mu=1}^r \epsilon_\mu w_k + \sum_{l=1}^h \sum_{\mu=1}^r \epsilon_\mu y_l} \right\} = H_{\mu,k,l} - \{(0, 0, 0, 0, 0)\}$$

$$V_1 = \left\{ V_{\sum_{\mu=1}^r m_\mu \epsilon_\mu + \sum_{i=1}^e \sum_{\mu=1}^r \epsilon_\mu u_i + \sum_{j=1}^f \sum_{\mu=1}^r \epsilon_\mu v_j + \sum_{k=1}^g \sum_{\mu=1}^r \epsilon_\mu w_k + \sum_{l=1}^h \sum_{\mu=1}^r \epsilon_\mu y_l} \right\} = Z(R)^* - \bigcup_{\mu,k,j} \left\{ V_{\sum_{\mu=1}^r m_\mu \epsilon_\mu + \sum_{k=1}^g \sum_{\mu=1}^r \epsilon_\mu w_k + \sum_{l=1}^h \sum_{\mu=1}^r \epsilon_\mu y_l} \right\}$$

From the definition of V_1 , we have the set of neighbours of V_1 as

$$N(V_1) = \bigcup_{\mu,k,j} \left\{ V_{\sum_{\mu=1}^r m_\mu \epsilon_\mu + \sum_{k=1}^g \sum_{\mu=1}^r \epsilon_\mu w_k + \sum_{l=1}^h \sum_{\mu=1}^r \epsilon_\mu y_l} \right\}$$

Clearly $|N(V_1)| = \left| \bigcup_{\mu,k,j} \left\{ V_{\sum_{\mu=1}^r m_\mu \epsilon_\mu + \sum_{k=1}^g \sum_{\mu=1}^r \epsilon_\mu w_k + \sum_{l=1}^h \sum_{\mu=1}^r \epsilon_\mu y_l} \right\} \right| = p^{(3+g+h)r} - 1$. Therefore

$$|V_1| = |Z(R)^*| - |N(V_1)| = p^{(3+e+f+g+h)r} - 1 - (p^{(3+g+h)r} - 1)$$



$$= p^{(3+g+h)r} (p^{(e+f)r} - 1)$$

Thus the binding number of the graph $\Gamma(R)$ is given by

$$b(\Gamma(R)) = \frac{|N(V_1)|}{V_1} = \frac{p^{(3+g+h)r} - 1}{p^{(3+g+h)r} (p^{(e+f)r} - 1)}$$

Since $N(V_1) \neq \Phi$ and $V_1 \cap N(V_1) = \Phi$ then the sets V_1 and $N(V_1)$ are mutually disjoint and as a result $Z(R^*) = V_1 \cup N(V_1)$. In addition $|\cup(N(V_1))| = p^{(3+g+h)r} - 1$ and hence $\Gamma(R)$ is $p^{(3+g+h)r}$ partite.

For $\zeta = 2$ let

$$H_{\mu,k,l} = \left\{ \sum_{\mu=1}^r m_{\mu} \epsilon_{\mu} + p \sum_{i=1}^e \sum_{\mu=1}^r \epsilon_{\mu} u_i + \sum_{k=1}^g \sum_{\mu=1}^r \epsilon_{\mu} w_k + \sum_{l=1}^h \sum_{\mu=1}^r \epsilon_{\mu} y_l, m_{\mu} \in \{0, \lambda(p)\} : 1 \leq \lambda \leq p^3 - 1 \right\}$$

Then the set $Z(R)^*$ is partitioned into the following mutually disjoint subsets;

$$\left\{ V_{\sum_{\mu=1}^r m_{\mu} \epsilon_{\mu} + p \sum_{i=1}^e \sum_{\mu=1}^r \epsilon_{\mu} u_i + \sum_{k=1}^g \sum_{\mu=1}^r \epsilon_{\mu} w_k + \sum_{l=1}^h \sum_{\mu=1}^r \epsilon_{\mu} y_l} \right\} = H_{\mu,k,l} - \{(0, 0, 0, 0)\}$$

$$V_1 = \left\{ V_{\sum_{\mu=1}^r m_{\mu} \epsilon_{\mu} + p \sum_{i=1}^e \sum_{\mu=1}^r \epsilon_{\mu} u_i + \sum_{j=1}^f \sum_{\mu=1}^r \epsilon_{\mu} v_j + \sum_{k=1}^g \sum_{\mu=1}^r \epsilon_{\mu} w_k + \sum_{l=1}^h \sum_{\mu=1}^r \epsilon_{\mu} y_l} \right\} = Z(R^*) - \bigcup_{\mu,k,j} \left\{ V_{\sum_{\mu=1}^r m_{\mu} \epsilon_{\mu} + p \sum_{i=1}^e \sum_{\mu=1}^r \epsilon_{\mu} u_i + \sum_{k=1}^g \sum_{\mu=1}^r \epsilon_{\mu} w_k + \sum_{l=1}^h \sum_{\mu=1}^r \epsilon_{\mu} y_l} \right\}$$

Clearly $|N(V_1)| = \left| \bigcup_{\mu,k,j} \left\{ V_{\sum_{\mu=1}^r m_{\mu} \epsilon_{\mu} + p \sum_{i=1}^e \sum_{\mu=1}^r \epsilon_{\mu} u_i + \sum_{k=1}^g \sum_{\mu=1}^r \epsilon_{\mu} w_k + \sum_{l=1}^h \sum_{\mu=1}^r \epsilon_{\mu} y_l} \right\} \right| = p^{(3+e+g+h)r} - 1$.
Therefore

$$\begin{aligned} |V_1| &= |Z(R)^*| - |N(V_1)| = p^{(3+2e+f+g+h)r} - 1 - (p^{3+e+g+h} - 1) \\ &= p^{(3+e+g+h)r} (p^{(e+f)r} - 1) \end{aligned}$$

Thus the binding number of the graph $\Gamma(R)$ is given by

$$b(\Gamma(R)) = \frac{|N(V_1)|}{V_1} = \frac{p^{(3+e+g+h)r} - 1}{p^{(3+e+g+h)r} (p^{(e+f)r} - 1)}$$

Since $N(V_1) \neq \Phi$ and $V_1 \cap N(V_1) = \Phi$ then the sets V_1 and $N(V_1)$ are mutually disjoint and as a result $Z(R^*) = V_1 \cup N(V_1)$. In addition $|\cup(N(V_1))| = p^{(3+e+g+h)r} - 1$ and hence $\Gamma(R)$ is $p^{(3+e+g+h)r}$ partite.

For $\zeta = 3$ $Z(R)^*$ is partitioned as in the case when $\zeta = 2$ but with

$$|N(V_1)| = \left| \bigcup_{\mu,k,j} \left\{ V_{\sum_{\mu=1}^r m_{\mu} \epsilon_{\mu} + p \sum_{i=1}^e \sum_{\mu=1}^r \epsilon_{\mu} u_i + \sum_{k=1}^g \sum_{\mu=1}^r \epsilon_{\mu} w_k + \sum_{l=1}^h \sum_{\mu=1}^r \epsilon_{\mu} y_l} \right\} \right| = p^{(3+2e+g+h)r} - 1 \text{ and}$$

$$\begin{aligned} |V_1| &= |Z(R)^*| - |N(V_1)| = p^{(3+3e+f+g+h)r} - 1 - (p^{3+2e+g+h} - 1) \\ &= p^{(3+2e+g+h)r} (p^{(e+f)r} - 1) \end{aligned}$$

Thus the binding number of the graph $\Gamma(R)$ is given by

$$b(\Gamma(R)) = \frac{|N(V_1)|}{V_1} = \frac{p^{(3+2e+g+h)r} - 1}{p^{(3+2e+g+h)r} (p^{(e+f)r} - 1)}$$



Since $N(V_1) \neq \Phi$ and $V_1 \cap N(V_1) = \Phi$ then the sets V_1 and $N(V_1)$ are mutually disjoint and as a result $Z(R^*) = V_1 \cup N(V_1)$. In addition $|\cup(N(V_1))| = p^{(3+2e+g+h)r} - 1$ and hence $\Gamma(R)$ is $p^{(3+2e+g+h)r}$ partite. For $\zeta = 4$ $Z(R)^*$ is partitioned as in the case when $\zeta = 2$ but with

$$|N(V_1)| = |\cup_{\mu,k,j} \{V_{\sum_{\mu=1}^r m_{\mu} \epsilon_{\mu} + p \sum_{i=1}^e \sum_{\mu=1}^r \epsilon_{\mu} u_i + \sum_{k=1}^g \sum_{\mu=1}^r \epsilon_{\mu} w_k + \sum_{l=1}^h \sum_{\mu=1}^r \epsilon_{\mu} y_l}\}| = p^{(3+3e+g+h)r} - 1 \text{ and}$$

$$\begin{aligned} |V_1| &= |Z(R)^*| - |N(V_1)| = p^{(3+4e+f+g+h)r} - 1 - (p^{3+3e+g+h} - 1) \\ &= p^{(3+3e+g+h)r} (p^{(e+f)r} - 1) \end{aligned}$$

Thus the binding number of the graph $\Gamma(R)$ is given by

$$b(\Gamma(R)) = \frac{|N(V_1)|}{|V_1|} = \frac{p^{(3+3e+g+h)r} - 1}{p^{(3+3e+g+h)r} (p^{(e+f)r} - 1)}$$

Since $N(V_1) \neq \Phi$ and $V_1 \cap N(V_1) = \Phi$ then the sets V_1 and $N(V_1)$ are mutually disjoint and as a result $Z(R^*) = V_1 \cup N(V_1)$. In addition $|\cup(N(V_1))| = p^{(3+3e+g+h)r} - 1$ and hence $\Gamma(R)$ is $p^{(3+3e+g+h)r}$ partite. □

4.5 Rings of characteristic p^5

The structures of zero divisors of this class of rings is given by:

- (i) $Z(R) = pR_0 \oplus U \oplus V \oplus W \oplus Y$
- (ii) $(Z(R))^2 = pR_0 \oplus pU \oplus V \oplus W \oplus Y$
- (iii) $(Z(R))^3 = p^2R_0 \oplus p^2U \oplus pV \oplus W \oplus Y$
- (iv) $(Z(R))^4 = p^3U \oplus p^2V \oplus pW \oplus Y$
- (v) $(Z(R))^5 = (0)$

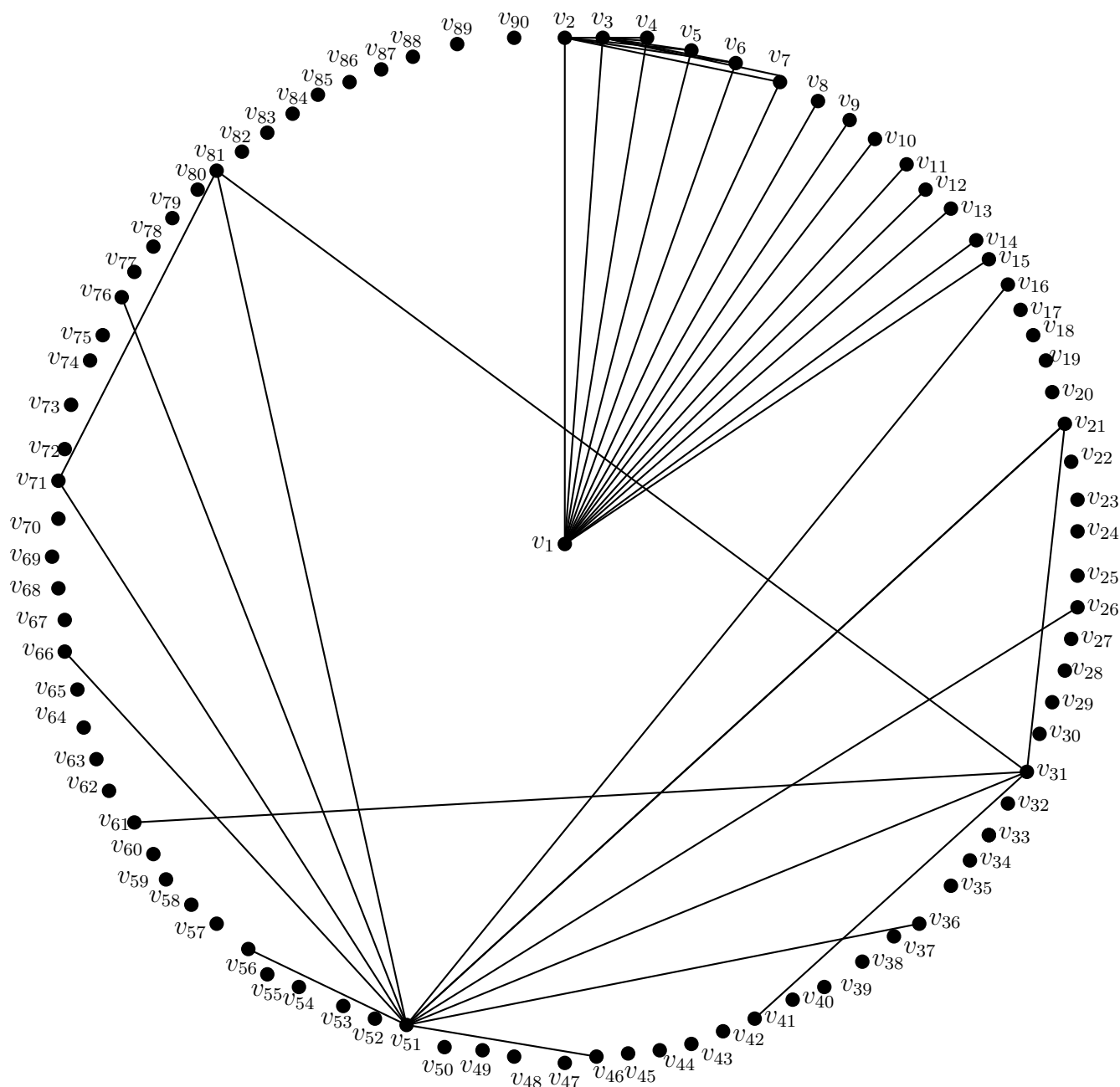
Given a zero divisor graph $\Gamma(R)$ with vertex set $V(\Gamma(R))$. If x and y are any two vertices of the graph, then x and y lie in the edge of the graph $E(\Gamma(R))$ if and only if $xy = 0$. Using this adjacency property, we have the following results:

Example 4.5. Let $Z(R) = 2\mathbb{Z}_{16} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus (Z)_2$. In this case $p = 2, r = 1, e = 1, f = 1, g = 1, h = 1$ and $h = 1$. Then

$$\Gamma(R) \text{ is } 64\text{-partite with } diam(\Gamma(R)) = 2, gr(\Gamma(R)) = 3 \text{ and } b(\Gamma(R)) = \frac{21}{64}.$$



When $\text{char}R = p^5$ the graph is as follows;



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Proposition 4.5. Let R be the ring constructed in [11] with $p^\zeta u_i = 0 \ \zeta = 1, 2, 3, 4; pv_j = pw_k = py_l = 0, 1 \leq i \leq e, 1 \leq j \leq f, 1 \leq k \leq g$ and $1 \leq l \leq h$. Then the graph of the ring $R, \Gamma(R)$ satisfies the following:

(i) $|V(\Gamma(R))| = p^{(3+\zeta e+f+g+h)r} - 1$

(ii) $b(\Gamma(R)) =$

$$\begin{cases} \frac{p^{(4+g+h)r}-1}{p^{(4+g+h)r}(p^{(e+f)r}-1)} & \text{if; } \zeta = 1 \\ \frac{p^{(4+e+g+h)r}-1}{(p^{(4+e+g+h)r}(p^{(e+f)r}-1))} & \text{if; } \zeta = 2 \\ \frac{p^{(4+2e+g+h)r}-1}{(p^{(4+2e+g+h)r}(p^{(e+f)r}-1))} & \text{if; } \zeta = 3 \\ \frac{p^{(4+3e+g+h)r}-1}{(p^{(4+3e+g+h)r}(p^{(e+f)r}-1))} & \text{if; } \zeta = 4. \end{cases}$$

(iii)

$$\Gamma(R) = \begin{cases} p^{(4+g+h)r} - \text{partite,} & \text{if; } \zeta = 1 \\ p^{(4+e+g+h)r} - \text{partite,} & \text{if; } \zeta = 2 \\ p^{(4+2e+g+h)r} - \text{partite,} & \text{if; } \zeta = 3 \\ p^{(4+3e+g+h)r} - \text{partite,} & \text{if; } \zeta = 4. \end{cases}$$

Proof. Since $Z(R) = pR_0 \oplus U \oplus V \oplus W \oplus Y$, we have $|Z(R)| = p^{(4+\zeta e+f+g+h)r}$ and because the set $Z(R)^* = Z(R) - \{(0, 0, 0, 0, 0)\}$, it follows that $|(Z(R))^*| = |V(\Gamma(R))| = p^{(4+\zeta e+f+g+h)r} - 1$ which establishes (i).

To establish (ii) and (iii), let $\epsilon_1, \dots, \epsilon_r \in R_0$ with $\epsilon_1 = 1$ such that $\bar{\epsilon}_1, \dots, \bar{\epsilon}_r \in R_0/pR_0$ form a basis for R_0/pR_0 regarded as a vector space over its prime subfield \mathbb{F}_p .

For $\zeta = 1$, let

$$H_{\mu,k,l} = \left\{ \sum_{\mu=1}^r m_\mu \epsilon_\mu + \sum_{k=1}^g \sum_{\mu=1}^r \epsilon_\mu w_k + \sum_{l=1}^h \sum_{\mu=1}^r \epsilon_\mu y_l, m_\mu \in \{0, (\lambda - 1)(p)\} : 2 \leq \lambda \leq p^4 \right\}$$

Then the set $Z(R)^*$ is partitioned into the following mutually disjoint subsets;

$$\left\{ V_{\sum_{\mu=1}^r m_\mu \epsilon_\mu + \sum_{k=1}^g \sum_{\mu=1}^r \epsilon_\mu w_k + \sum_{l=1}^h \sum_{\mu=1}^r \epsilon_\mu y_l} \right\} = H_{\mu,k,l} - \{(0, 0, 0, 0, 0)\}$$

$$V_1 = \left\{ V_{\sum_{\mu=1}^r m_\mu \epsilon_\mu + \sum_{i=1}^e \sum_{\mu=1}^r \epsilon_\mu u_i + \sum_{j=1}^f \sum_{\mu=1}^r \epsilon_\mu v_j + \sum_{k=1}^g \sum_{\mu=1}^r \epsilon_\mu w_k + \sum_{l=1}^h \sum_{\mu=1}^r \epsilon_\mu y_l} \right\} = Z(R)^* - \bigcup_{\mu,k,j} \left\{ V_{\sum_{\mu=1}^r m_\mu \epsilon_\mu + \sum_{k=1}^g \sum_{\mu=1}^r \epsilon_\mu w_k + \sum_{l=1}^h \sum_{\mu=1}^r \epsilon_\mu y_l} \right\}$$

From the definition of V_1 , we have the set of neighbours of V_1 as

$$N(V_1) = \bigcup_{\mu,k,j} \left\{ V_{\sum_{\mu=1}^r m_\mu \epsilon_\mu + \sum_{k=1}^g \sum_{\mu=1}^r \epsilon_\mu w_k + \sum_{l=1}^h \sum_{\mu=1}^r \epsilon_\mu y_l} \right\}$$

Clearly $|N(V_1)| = \left| \bigcup_{\mu,k,j} \left\{ V_{\sum_{\mu=1}^r m_\mu \epsilon_\mu + \sum_{k=1}^g \sum_{\mu=1}^r \epsilon_\mu w_k + \sum_{l=1}^h \sum_{\mu=1}^r \epsilon_\mu y_l} \right\} \right| = p^{(4+g+h)r} - 1$. Therefore

$$|V_1| = |Z(R)^*| - |N(V_1)| = p^{(4+e+f+g+h)r} - 1 - (p^{(4+g+h)r} - 1)$$



$$= p^{(4+g+h)r} (p^{(e+f)r} - 1)$$

Thus the binding number of the graph $\Gamma(R)$ is given by

$$b(\Gamma(R)) = \frac{|N(V_1)|}{V_1} = \frac{p^{(4+g+h)r} - 1}{p^{(4+g+h)r} (p^{(e+f)r} - 1)}$$

Since $N(V_1) \neq \Phi$ and $V_1 \cap N(V_1) = \Phi$ then the sets V_1 and $N(V_1)$ are mutually disjoint and as a result $Z(R^*) = V_1 \cup N(V_1)$. In addition $|\cup(N(V_1))| = p^{(4+g+h)r} - 1$ and hence $\Gamma(R)$ is $p^{(4+g+h)r}$ partite.

For $\zeta = 2$ let

$$H_{\mu,i,k,l} = \left\{ \sum_{\mu=1}^r m_{\mu} \epsilon_{\mu} + p \sum_{i=1}^e \sum_{\mu=1}^r \epsilon_{\mu} u_i + \sum_{k=1}^g \sum_{\mu=1}^r \epsilon_{\mu} w_k + \sum_{l=1}^h \sum_{\mu=1}^r \epsilon_{\mu} y_l, m_{\mu} \in \{0, (\lambda - 1)(p)\} : 2 \leq \lambda \leq p^4 \right\}$$

Then the set $Z(R)^*$ is partitioned into the following mutually disjoint subsets;

$$\left\{ V_{\sum_{\mu=1}^r m_{\mu} \epsilon_{\mu} + p \sum_{i=1}^e \sum_{\mu=1}^r \epsilon_{\mu} u_i + \sum_{k=1}^g \sum_{\mu=1}^r \epsilon_{\mu} w_k + \sum_{l=1}^h \sum_{\mu=1}^r \epsilon_{\mu} y_l} \right\} = H_{\mu,i,k,l} - \{(0, 0, 0, 0)\}$$

$$V_1 = \left\{ V_{\sum_{\mu=1}^r m_{\mu} \epsilon_{\mu} + p \sum_{i=1}^e \sum_{\mu=1}^r \epsilon_{\mu} u_i + \sum_{j=1}^f \sum_{\mu=1}^r \epsilon_{\mu} v_j + \sum_{k=1}^g \sum_{\mu=1}^r \epsilon_{\mu} w_k + \sum_{l=1}^h \sum_{\mu=1}^r \epsilon_{\mu} y_l} \right\} = Z(R^*) - \bigcup_{\mu,i,k,j} \left\{ V_{\sum_{\mu=1}^r m_{\mu} \epsilon_{\mu} + p \sum_{i=1}^e \sum_{\mu=1}^r \epsilon_{\mu} u_i + \sum_{k=1}^g \sum_{\mu=1}^r \epsilon_{\mu} w_k + \sum_{l=1}^h \sum_{\mu=1}^r \epsilon_{\mu} y_l} \right\}$$

From the definition of V_1 , we have the set of neighbors of V_1 as

$$N(V_1) = \bigcup_{\mu,i,k,j} \left\{ V_{\sum_{\mu=1}^r m_{\mu} \epsilon_{\mu} + p \sum_{i=1}^e \sum_{\mu=1}^r \epsilon_{\mu} u_i + \sum_{k=1}^g \sum_{\mu=1}^r \epsilon_{\mu} w_k + \sum_{l=1}^h \sum_{\mu=1}^r \epsilon_{\mu} y_l} \right\}$$

Clearly $|N(V_1)| = \left| \bigcup_{\mu,i,k,j} \left\{ V_{\sum_{\mu=1}^r m_{\mu} \epsilon_{\mu} + p \sum_{i=1}^e \sum_{\mu=1}^r \epsilon_{\mu} u_i + \sum_{k=1}^g \sum_{\mu=1}^r \epsilon_{\mu} w_k + \sum_{l=1}^h \sum_{\mu=1}^r \epsilon_{\mu} y_l} \right\} \right| = p^{(4+e+g+h)r} - 1$. Therefore

$$|V_1| = |Z(R)^*| - |N(V_1)| = p^{(4+2e+f+g+h)r} - 1 - (p^{4+e+g+h} - 1)$$

$$= p^{(4+e+g+h)r} (p^{(e+f)r} - 1)$$

Thus the binding number of the graph $\Gamma(R)$ is given by

$$b(\Gamma(R)) = \frac{|N(V_1)|}{V_1} = \frac{p^{(4+e+g+h)r} - 1}{p^{(4+e+g+h)r} (p^{(e+f)r} - 1)}$$

Since $N(V_1) \neq \Phi$ and $V_1 \cap N(V_1) = \Phi$ then the sets V_1 and $N(V_1)$ are mutually disjoint and as a result $Z(R^*) = V_1 \cup N(V_1)$. In addition $|\cup(N(V_1))| = p^{(4+e+g+h)r} - 1$ and hence $\Gamma(R)$ is $p^{(4+e+g+h)r}$ partite.

For $\zeta = 3$ $Z(R)^*$ is partitioned as in the case when $\zeta = 2$ but with

$$|N(V_1)| = \left| \bigcup_{\mu,i,k,j} \left\{ V_{\sum_{\mu=1}^r m_{\mu} \epsilon_{\mu} + p \sum_{i=1}^e \sum_{\mu=1}^r \epsilon_{\mu} u_i + \sum_{k=1}^g \sum_{\mu=1}^r \epsilon_{\mu} w_k + \sum_{l=1}^h \sum_{\mu=1}^r \epsilon_{\mu} y_l} \right\} \right| = p^{(4+2e+g+h)r} - 1$$
 and

$$|V_1| = |Z(R)^*| - |N(V_1)| = p^{(4+3e+f+g+h)r} - 1 - (p^{4+2e+g+h} - 1)$$



$$= p^{(4+2e+g+h)r} (p^{(e+f)r} - 1)$$

Thus the binding number of the graph $\Gamma(R)$ is given by

$$b(\Gamma(R)) = \frac{|N(V_1)|}{V_1} = \frac{p^{(4+2e+g+h)r} - 1}{p^{(4+2e+g+h)r} (p^{(e+f)r} - 1)}$$

Since $N(V_1) \neq \Phi$ and $V_1 \cap N(V_1) = \Phi$ then the sets V_1 and $N(V_1)$ are mutually disjoint and as a result $Z(R^*) = V_1 \cup N(V_1)$. In addition $|\cup(N(V_1))| = p^{(4+2e+g+h)r} - 1$ and hence $\Gamma(R)$ is $p^{(4+2e+g+h)r}$ partite.

For $\zeta = 4$ $Z(R)^*$ is partitioned as in the case when $\zeta = 2$ but with $|N(V_1)| = |\cup_{\mu,i,k,j} \{V_{\sum_{\mu=1}^r m_{\mu} \epsilon_{\mu} + p \sum_{i=1}^e \sum_{\mu=1}^r \epsilon_{\mu} u_i + \sum_{k=1}^g \sum_{\mu=1}^r \epsilon_{\mu} w_k + \sum_{l=1}^h \sum_{\mu=1}^r \epsilon_{\mu} y_l}\}| = p^{(4+3e+g+h)r} - 1$ and

$$\begin{aligned} |V_1| &= |Z(R)^*| - |N(V_1)| = p^{(4+4e+f+g+h)r} - 1 - (p^{(4+3e+g+h)r} - 1) \\ &= p^{(4+3e+g+h)r} (p^{(e+f)r} - 1) \end{aligned}$$

Thus the binding number of the graph $\Gamma(R)$ is given by

$$b(\Gamma(R)) = \frac{|N(V_1)|}{V_1} = \frac{p^{(4+3e+g+h)r} - 1}{p^{(4+3e+g+h)r} (p^{(e+f)r} - 1)}$$

Since $N(V_1) \neq \Phi$ and $V_1 \cap N(V_1) = \Phi$ then the sets V_1 and $N(V_1)$ are mutually disjoint and as a result $Z(R^*) = V_1 \cup N(V_1)$. In addition $|\cup(N(V_1))| = p^{(4+3e+g+h)r} - 1$ and hence $\Gamma(R)$ is $p^{(4+3e+g+h)r}$ partite.

5 Conclusion

In conclusion, the paper provides a survey on the properties of the zero divisors $Z(R)$, the graphs $\Gamma(R)$ for idealized commutative rings. In particular, the binding numbers obtained demonstrate geometric distribution of graph edge set $E(\Gamma(R))$ from a common vertex $v \in V(\Gamma(R))$. □

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