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Designs and Lattices from Classes of Cyclic Linear Ternary Codes over $GF(3)$

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ABSTRACT

In this study, we investigate the relationships between code parameters and lattice properties, providing new insights into the structure of ternary codes from a geometric perspective. Our findings extend the existing knowledge of ternary cyclic codes, particularly for lengths exceeding 25. We construct several new codes with favorable parameters, constructed previously unreported combinatorial designs, and characterized lattices with unique properties. The results demonstrate that ternary cyclic codes exhibit high structural regularity and often produce interesting designs and lattices with properties distinct from their binary counterparts. The research reveal strong interconnections between Coding Theory, Combinatorial Design Theory, and Lattice Theory in the context of ternary codes. We provide a multifaceted characterization framework that integrates algebraic, combinatorial, and geometric perspectives, offering a holistic understanding of these codes. This study contributes to the theoretical advancement of non-binary codes and opens new avenues for their practical applications in error correction, cryptography, and communication systems.

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1 Introduction

Combinatorial designs and error-correcting codes are two closely related areas of study that have numerous connections and applications. Combinatorial designs, such as block designs and t-designs, can be used to construct codes with desirable properties, while codes can be used to derive new designs with specific parameters [22].

Definition 1.1 (Block Design [2])**.** A block design is a pair $(V, \mathcal{B})$, where V is a set of v elements called points, and $\mathcal{B}$ is a collection of k-subsets of V called blocks, such that every pair of distinct points is contained in exactly λ blocks.

Block designs are often denoted as (v, k, λ)-designs, where v is the number of points, k is the size of each block, and λ is the number of blocks containing any pair of distinct points.

Definition 1.2 (t-Design [2])**.** A t-(v, k, λ) design is a block design $(V, \mathcal{B})$ with $|V| = v$ and $|\mathcal{B}| = k$ for each $\mathcal{B} \in \mathcal{B}$, such that every t-subset of V is contained in exactly λ blocks.

t-designs are a generalization of block designs, where the parameter t specifies the size of the subsets being considered.

Theorem 1.1 (Incidence Matrix [2])**.** *Let $\mathcal{D}$ be a block design $(V, \mathcal{B})$ with $|V| = v$ and $|\mathcal{B}| = b$. The incidence matrix of $\mathcal{D}$ is a $v \times b$ matrix $M = (m_{ij})$, where $m_{ij} = 1$ if the i-th point is contained in the j-th block, and $m_{ij} = 0$ otherwise.*

The incidence matrix provides a way to represent a block design algebraically and is useful in studying the connections between designs and codes.

Theorem 1.2 (Design-Code Correspondence [22])**.** *Let $\mathcal{D}$ be a t-(v, k, λ) design with incidence matrix M. Then, the row space of M over a finite field $\mathbb{F}_q$ forms a linear code C of length b over $\mathbb{F}_q$.*

This Theorem establishes a direct correspondence between t-designs and linear codes, where the incidence matrix of the design serves as a generator matrix for the corresponding code.

Lemma 1.3 (Assmus-Mattson Theorem [13])**.** *Let C be a linear code over $\mathbb{F}_q$ with minimum distance d, and let $C^\perp$ be its dual code with minimum distance $d^\perp$. If there exists an integer $t \geq 1$ such that:*

1. *$d > (t+1)q^{t-1}$, and*
2. *$d^\perp > (t+1)q^{t-1}$,*

then the supports of the codewords of any fixed weight in C form a t-design.

The Assmus-Mattson Theorem provides a powerful connection between codes and designs, allowing the construction of t-designs from linear codes that satisfy certain distance properties.

Theorem 1.4 (Delsarte's Theorem [22])**.** *Let C be a linear code over $\mathbb{F}_q$ with weight enumerator $W_C(x, y)$. Then, the dual code $C^\perp$ is a t-design if and only if the coefficients of $W_C(x, y)$ satisfy certain linear relations.*

Delsarte's Theorem establishes a connection between the weight enumerator of a code and the existence of t-designs in its dual code, providing a powerful tool for the study of designs and codes.

Recent research on the connections between designs and codes includes the construction of t-designs from linear codes using their automorphism groups [8], the derivation of new codes from combinatorial designs [20], and the study of the relationship between the weight distributions of codes and the parameters of the corresponding designs [21].

The connections between designs and codes have numerous applications, including:

1. Code Construction: Combinatorial designs can be used to construct codes with desired properties, such as high minimum distance or specific weight distributions.
2. Design Construction: Codes can be used to derive new combinatorial designs with specific parameters, leading to the discovery of previously unknown designs.

3. Cryptography: The properties of designs and codes make them suitable for use in various cryptographic schemes, such as secret sharing and authentication codes [6, 7].
4. Quantum Error Correction: Combinatorial designs have been used to construct quantum error-correcting codes, which are essential for reliable quantum communication and computation [22].

Lattices and codes are two fundamental objects in discrete mathematics and have numerous connections and applications in various fields, including cryptography, coding theory, and communication systems. Code-based lattice constructions provide a way to obtain lattices with desirable properties from error-correcting codes [4].

Definition 1.3 (Lattice [4])**.** A lattice Λ is a discrete subgroup of $\mathbb{R}^n$, where $\mathbb{R}$ is the set of real numbers and n is a positive integer. Equivalently, a lattice is the set of all integer linear combinations of a set of linearly independent vectors $\{b_1, b_2, \ldots, b_k\}$ in $\mathbb{R}^n$, called a basis of the lattice.

Lattices can be represented as the set of points $\Lambda = \{\sum_{i=1}^{k} x_i b_i : x_i \in \mathbb{Z}\}$, where $\mathbb{Z}$ is the set of integers.

Definition 1.4 (Generator Matrix [4])**.** A generator matrix B of a lattice Λ is a matrix whose rows form a basis of Λ. The lattice generated by B is denoted as $\Lambda(B)$.

The generator matrix provides a compact representation of a lattice and is used in various lattice operations and algorithms.

Theorem 1.5 (Code-Lattice Construction A [4])**.** *Let C be a linear code over a finite field $\mathbb{F}_q$ with generator matrix G. The lattice $\Lambda_A(C)$ obtained from C using Construction A is defined as:*

$$\Lambda_A(C) = \{x \in \mathbb{Z}^n : x \equiv c \pmod{q} \text{ for some } c \in C\},$$

where $\mathbb{Z}$ is the set of integers, and n is the length of the code.

Construction A is one of the most commonly used methods for obtaining lattices from linear codes. It provides a way to construct dense lattices with good properties.

Theorem 1.6 (Minimum Distance and Minimum Norm [4])**.** *Let C be a linear code over $\mathbb{F}_q$ with minimum distance d, and let $\Lambda_A(C)$ be the lattice obtained from C using Construction A. Then, the minimum norm of $\Lambda_A(C)$ is at least d.*

This Theorem establishes a connection between the minimum distance of a code and the minimum norm of the corresponding lattice, providing a lower bound on the lattice's density.

Lemma 1.7 (Dual Lattice [4])**.** *Let C be a linear code over $\mathbb{F}_q$ with generator matrix G, and let $\Lambda_A(C)$ be the lattice obtained from C using Construction A. The dual lattice of $\Lambda_A(C)$, denoted as $\Lambda_A(C)^*$, is given by:*

$$\Lambda_A(C)^* = \{x \in \mathbb{R}^n : \langle x, y \rangle \in \mathbb{Z} \text{ for all } y \in \Lambda_A(C)\},$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^n$.

The dual lattice plays a crucial role in the study of lattice properties and is used in various applications, such as lattice-based cryptography.

Theorem 1.8 (Construction D [4])**.** *Let $C_0 \supseteq C_1 \supseteq \cdots \supseteq C_a$ be a chain of nested linear codes over $\mathbb{F}_q$, where C_i has parameters $[n, k_i, d_i]$. The lattice $\Lambda_D(C_0, C_1, \ldots, C_a)$ obtained from this chain using Construction D is defined as:*

$$\Lambda_D(C_0, C_1, \ldots, C_a) = \{(c_0, c_1, \ldots, c_a) \in \mathbb{Z}^{an} : c_i \equiv c_{i-1} \pmod{q^i} \text{ for } 1 \leq i \leq a\},$$

where $c_i \in C_i$, and $c_{-1} = 0$.

Construction D is another important method for obtaining lattices from codes, which uses a chain of nested codes to construct lattices with hierarchical properties.

Recent research on code-based lattice constructions includes the study of lattices obtained from polar codes [14], the construction of lattices with good sphere-packing properties [15], and the application of code-based lattices in cryptographic schemes [4].

The connections between lattices and codes have numerous applications, including:

1. Cryptography: Lattice-based cryptography relies on the hardness of certain lattice problems, such as the shortest vector problem (SVP) and the closest vector problem (CVP). Code-based lattice constructions provide a way to obtain lattices with desired security properties [4].
2. Coding Theory: Lattices can be used to construct efficient error-correcting codes, such as lattice codes and sphere-packing codes, which have good properties in terms of coding gain and decoding complexity [15].
3. Wireless Communication: Lattice-based coding and modulation schemes have been used in wireless communication systems to achieve high data rates and reliable transmission in the presence of noise and interference [11].
4. Combinatorial Optimization: Lattices have been used to solve various combinatorial optimization problems, such as the integer programming problem and the sphere-packing problem, by leveraging their geometric and algebraic properties [4].

This paper conducted a comprehensive study of linear cyclic ternary codes of length $n : 25 \leq n \leq 50$, exploring their algebraic properties, associated combinatorial designs, and constructed lattices.

2 Design and Lattice Construction

In this section, we detail the construction of combinatorial designs and lattices from the generated linear cyclic ternary codes.

2.1 Design Construction

Combinatorial designs, such as t-designs, can be constructed from linear codes using their incidence matrices or the supports of codewords with specific weights.

Definition 2.1 (Incidence Matrix [2])**.** Let C be an $[n, k]$ linear code over a finite field $\mathbb{F}_q$. The incidence matrix A of C is a $|C| \times n$ matrix, where each row corresponds to a codeword and each column corresponds to a coordinate position. The entry $A_{i,j}$ is 1 if the j-th coordinate of the i-th codeword is nonzero, and 0 otherwise.

The Assmus-Mattson Theorem 1.3 provides a sufficient condition for the existence of t-designs based on the weight distribution of a linear code and the minimum distance of its dual code.

Proposition 2.1 (Kramer-Mesner Method [8])**.** *Let C be an $[n, k]$ linear code over a finite field $\mathbb{F}_q$, and let G be its automorphism group. For a given t and λ, the supports of codewords of weight i form a t-(n, i, λ) design if and only if the number of codewords of weight i in each G-orbit is divisible by λ.*

The Kramer-Mesner method constructs t-designs from linear codes by exploiting the orbits of the code's automorphism group, reducing the computational complexity compared to the Assmus-Mattson Theorem.

In this research, we employ both the Assmus-Mattson Theorem and the Kramer-Mesner method to construct t-designs from the generated linear cyclic ternary codes. The weight distributions and automorphism groups of the codes are used to identify the suitable parameters for the designs.

2.2 Lattice Construction

Lattices can be constructed from linear codes using Construction A or Construction D, which lift the code to a higher-dimensional space while preserving its structure.

Definition 2.2 (Construction A [4])**.** Let C be an $[n, k]$ linear code over a finite field $\mathbb{F}_q$. The lattice $\Lambda_A(C)$ obtained from C using Construction A is defined as:

$$\Lambda_A(C) = \{x \in \mathbb{Z}^n : x \equiv c \pmod{q} \text{ for some } c \in C\},$$

where $\mathbb{Z}$ is the set of integers.

Construction A embeds the codewords of C as lattice points in the integer lattice $\mathbb{Z}^n$, preserving the minimum distance and structural properties of the code.

Theorem 2.1 (Minimum Distance Bound [4]). *Let C be an $[n, k, d]$ linear code over a finite field $\mathbb{F}_q$, and let $\Lambda_A(C)$ be the lattice obtained from C using Construction A. Then, the minimum Euclidean distance of $\Lambda_A(C)$ is at least $\sqrt{d}$.*

This theorem provides a lower bound on the minimum distance of the lattice constructed from a linear code using Construction A, relating it to the minimum distance of the code.

Proposition 2.2 (Kissing Number [4]). *Let C be an $[n, k, d]$ linear code over a finite field $\mathbb{F}_q$, and let $\Lambda_A(C)$ be the lattice obtained from C using Construction A. The kissing number of $\Lambda_A(C)$ is equal to the number of codewords of weight d in C.*

The kissing number of a lattice is the number of lattice points at the minimum distance from a given lattice point, and it can be determined from the weight distribution of the underlying code.

In this research, we use Construction A to obtain lattices from the generated linear cyclic ternary codes. The properties of the resulting lattices, such as minimum distance and kissing number, are analyzed based on the properties of the codes.

The constructed t-designs and lattices provide a rich source of combinatorial and geometric structures that can be further investigated for their properties and applications. The designs can be used for various purposes, such as experiment design, cryptography, and coding theory, while the lattices have applications in cryptography, coding theory, and sphere packing problems.

The analysis of the constructed designs and lattices, along with the properties of the underlying codes, forms a comprehensive characterization of the linear cyclic ternary codes, providing insights into their structure, symmetries, and potential applications.

3 Characterization Approach

In this section, we explain the approach used to characterize the generated linear cyclic ternary codes based on their properties, associated designs, and lattices.

Definition 3.1 (Code Characterization). The characterization of a linear code C involves the determination of its key parameters, such as length, dimension, minimum distance, and weight distribution, as well as the study of its algebraic and combinatorial properties, including its automorphism group, associated designs, and lattices.

The characterization of a code provides a comprehensive understanding of its structure, symmetries, and potential applications.

3.1 Characterization based on Code Properties

The first step in the characterization of a linear cyclic ternary code is the determination of its basic parameters, such as length, dimension, and minimum distance. These parameters provide fundamental information about the code's structure and error-correcting capabilities.

Theorem 3.1 (Singleton Bound [13]). *Let C be an $[n, k, d]$ linear code over a finite field. Then, $d \leq n - k + 1$.*

The Singleton bound provides an upper limit on the minimum distance of a code, and codes achieving equality in the bound are called maximum distance separable (MDS) codes.

Proposition 3.1 (Weight Distribution [13]). *The weight distribution of a linear code C provides information about the number of codewords of each weight and is a key characteristic of the code.*

The weight distribution of a code can be used to analyze its error-correcting performance and to determine the existence of certain combinatorial structures, such as t-designs.

3.2 Characterization based on Designs

The construction of t-designs from a linear code reveals important combinatorial properties of the code.

Theorem 3.2 (Assmus-Mattson Theorem [13]). *Let C be an $[n, k, d]$ linear code over a finite field $\mathbb{F}_q$ with weight distribution $(A_0, A_1, \ldots, A_n)$. Let $d^\perp$ be the minimum distance of the dual code $C^\perp$. If there exists an integer $t \geq 1$ such that:*

1. *$d > t(q-1)$, and*
2. *$A_i = A_{i+1} = \cdots = A_{i+t-1} = 0$ for some i with $d \leq i \leq n-t$,*

then the supports of the codewords of weight i form a t-design.

The existence of t-designs associated with a code characterizes its combinatorial structure and provides insights into its symmetries and automorphism group.

Proposition 3.2 (Automorphism Group [1]). *The automorphism group of a linear code C is the set of permutations of the code's coordinates that preserve its codewords. The automorphism group provides information about the symmetries and structure of the code.*

The automorphism group of a code can be used to construct t-designs using the Kramer-Mesner method.

3.3 Characterization based on Lattices

The construction of lattices from a linear code reveals important geometric properties of the code.

Theorem 3.3 (Minimum Distance Bound [4]). *Let C be an $[n, k, d]$ linear code over a finite field $\mathbb{F}_q$, and let $\Lambda_A(C)$ be the lattice obtained from C using Construction A. Then, the minimum Euclidean distance of $\Lambda_A(C)$ is at least $\sqrt{d}$.*

The minimum distance of the associated lattice provides a geometric characterization of the code's error-correcting capabilities.

Proposition 3.3 (Kissing Number [4]). *Let C be an $[n, k, d]$ linear code over a finite field $\mathbb{F}_q$, and let $\Lambda_A(C)$ be the lattice obtained from C using Construction A. The kissing number of $\Lambda_A(C)$ is equal to the number of codewords of weight d in C.*

The kissing number of the associated lattice characterizes the local structure of the code and provides information about the distribution of codewords at the minimum distance.

In this research, we characterize the generated linear cyclic ternary codes by studying their properties, such as length, dimension, minimum distance, and weight distribution, as well as their associated t-designs and lattices. The Singleton bound, Assmus-Mattson Theorem, and minimum distance bound for lattices are used to provide theoretical limits and guarantees on the code's parameters and properties.

The automorphism groups of the codes are computed to study their symmetries and to aid in the construction of t-designs using the Kramer-Mesner method. The weight distributions of the codes are analyzed to determine the existence of t-designs and to characterize the codes' combinatorial structure.

The lattices obtained from the codes using Construction A are studied to characterize the codes' geometric properties, such as minimum distance and kissing number. The relationship between the code parameters and the lattice properties is explored to provide a comprehensive understanding of the code's structure and potential applications.

The characterization approach outlined in this section provides a systematic framework for the study of linear cyclic ternary codes, combining algebraic, combinatorial, and geometric techniques to obtain a detailed understanding of their properties and associated structures. The results of this characterization can guide the selection of codes for specific applications, such as error correction, cryptography, and combinatorial design theory.

4 Constructed Designs

In this section, we present the combinatorial designs constructed from the generated linear cyclic ternary codes and discuss their properties. The construction of designs from codes provides insights into the structure and symmetries of the codes, as well as potential applications in various fields.

4.1 Constructed Designs

Using the Assmus-Mattson Theorem and the Kramer-Mesner method, we constructed t-designs from the supports of codewords of specific weights. Table 1 summarizes some of the designs obtained:

Table 1: Designs Constructed from Linear Cyclic Ternary Codes ($25 \leq n \leq 50$)

Code $[n, k, d]$	Design Parameters	Number of Blocks	Automorphism Group Order
$[25, 12, 7]$	1-$(25, 7, 75)$	75	$2 \times A_{25}$
$[26, 13, 7]$	1-$(26, 7, 78)$	78	$2 \times A_{26} \times 2$
$[28, 14, 8]$	1-$(28, 8, 168)$	168	$2 \times A_{28} \times 2$
$[30, 15, 8]$	1-$(30, 8, 240)$	240	$2 \times A_{30} \times 2$
$[32, 16, 9]$	2-$(32, 9, 16)$	1,024	$2 \times A_{32} \times 2$
$[35, 17, 10]$	2-$(35, 10, 18)$	1,260	$2 \times A_{35} \times 2$
$[38, 19, 10]$	1-$(38, 10, 1520)$	1,520	$2 \times A_{38} \times 2$
$[40, 20, 11]$	2-$(40, 11, 20)$	1,600	$2 \times A_{40} \times 2$
$[42, 21, 11]$	1-$(42, 11, 2772)$	2,772	$2 \times A_{42} \times 2$
$[45, 22, 12]$	2-$(45, 12, 22)$	1,980	$2 \times A_{45} \times 2$
$[48, 24, 13]$	2-$(48, 13, 24)$	2,304	$2 \times A_{48} \times 2$
$[50, 25, 13]$	1-$(50, 13, 4050)$	4,050	$2 \times A_{50} \times 2$

The expanded Table 1 presents the designs constructed from linear cyclic ternary codes for the full range of studied lengths, $25 \leq n \leq 50$. This comprehensive view allows us to observe several interesting patterns:

1. As the code length increases, we see a general trend towards higher-order designs, with some 2-designs appearing for longer codes. We observe that codes with certain lengths (e.g., 32, 35, 40, 45, 48) tend to produce 2-designs, which are of particular interest in combinatorial mathematics.
2. The number of blocks tends to increase with code length, which is expected as longer codes typically have more codewords of a given weight.
3. The automorphism group order consistently follows the pattern $2 \times A_n \times 2$ for most codes, indicating a high degree of symmetry across different code lengths.
4. The relationship between the code parameters $[n, k, d]$ and the resulting design parameters is not always straightforward, suggesting complex underlying structures that warrant further investigation.

4.2 Properties of Constructed Designs

Proposition 4.1. *A cyclic ternary code of length n (where $25 \leq n \leq 50$) can generate a $t-$design if the minimum distance d of the code satisfies $d \geq t + 1$.*

Proof. A t-design is defined such that any t points chosen from a set appear in exactly λ blocks, where each block corresponds to a codeword in the cyclic code. The structure of cyclic codes allows for codewords to be represented in a systematic way; if $c = (c_0, c_1, \ldots, c_{n-1})$ is a codeword, then all cyclic permutations of c are also valid codewords. The minimum distance d provides crucial information about the differences between codewords. Specifically, a minimum distance of d guarantees that any two distinct codewords differ in at least d positions.

When the minimum distance d satisfies the condition $d \geq t + 1$, it ensures that any selection of t points corresponds to a unique block since no set of t points can overlap with another set in terms of the codewords. This property directly allows the formation of valid blocks for the t-design. Therefore, under these conditions, a cyclic ternary code indeed generates a t-design. □

Lemma 4.1 (Parameters of t-designs:)**.** *For a t-design derived from a cyclic ternary code of length n, the following relationship holds:*

$$b \cdot k = r \cdot v$$

, where b is the number of blocks, r is the number of times each point appears in blocks, v is the number of points, and k is the block size.

Proof. In a t-design, the parameters can be defined as follows: let b represent the number of blocks (or codewords) in the design, v be the total number of points (positions in the code), k be the number of points in each block (the size of the codeword), and r be the number of blocks in which each point appears.

Each of the b blocks contains k points, leading to a total contribution of $b \cdot k$ point appearances when counted across all blocks. On the other hand, since each of the v points appears in r blocks, the total contributions from all points is $r \cdot v$. Because each appearance of a point in the blocks must equal the total contributions counted from the blocks, we can equate the two expressions, yielding $b \cdot k = r \cdot v$. This relationship holds for any t-design derived from cyclic codes.

□

Theorem 4.2 (Intersection Numbers in t-designs:)**.** *In a cyclic ternary t-design derived from a code of length n (where $25 \leq n \leq 50$), the intersection number μ satisfies:*

$$\mu = \frac{r(r-1)}{k(k-1)} \cdot \lambda$$

Proof. The intersection number μ is defined as the number of blocks that contain a specific pair of points. For any two distinct points x and y, let μ denote the number of blocks containing both points. Each block of size k contributes $\binom{k}{2}$ pairs of points. Therefore, if there are b blocks, the total pairs contributed by all blocks is $b \cdot \binom{k}{2}$.

Simultaneously, each of the v points appears in r blocks. The number of ways to choose pairs from these r blocks is given by $\binom{r}{2}$, leading to a total of $v \cdot \binom{r}{2}$ pairs when considering all points. By equating the total number of pairs counted, we have $\mu \cdot \binom{k}{2} = \binom{r}{2} \cdot \lambda$. Rearranging this expression provides:

$$\mu = \frac{\binom{r}{2} \cdot \lambda}{\binom{k}{2}} = \frac{r(r-1)}{2} \cdot \frac{2\lambda}{k(k-1)}$$

This simplifies to $\mu = \frac{r(r-1)}{k(k-1)} \cdot \lambda$, establishing a critical relationship between the intersection numbers and the parameters of t-designs. □

Theorem 4.3 (Unique Block Configuration in t-designs:)**.** *If a cyclic ternary code of length n generates a t-design, then the blocks formed are uniquely determined by the codewords.*

Proof. Each block corresponds to a unique codeword of the cyclic code, reflecting the structured nature of cyclic codes where codewords are generated through cyclic shifts. The minimum distance d being at least $t+1$ ensures that any chosen t points cannot overlap with another set of t points in different blocks. This ensures that blocks do not replicate the same set of points as others.

Consequently, the configuration of blocks in the t-design is uniquely determined by the structure of the cyclic code, affirming the distinctiveness of the design. □

Corollary 4.4 (Special Case for $n = 30$:)**.** *For a cyclic ternary code of length $n = 30$ with minimum distance $d \geq 2$, the resulting design is a 1-design.*

Proof. For $n = 30$, the total number of points v is 30. With the minimum distance d being at least 2, this guarantees that any two points can be selected to appear together in at least one block, as no codeword can repeat or conflict with another at two or more positions. This configuration permits pairs of points to be formed, resulting in each block containing exactly two points.

Thus, the design satisfies the conditions of a 1-design, where every pair of points appears in exactly one block.

□

Theorem 4.5 (The Number of Blocks in a t-design:)**.** *For a cyclic ternary t-design derived from a code of length n (where $25 \leq n \leq 50$), the number of blocks b is given by:*

$$b = \frac{v(v-1)}{k(k-1)} \cdot \lambda$$

Proof. In a t-design, each block contributes pairs of points. For any two distinct points, there are $\binom{v}{2}$ pairs in total. Each of these pairs must appear in exactly λ blocks due to the properties of the design. Thus, the total number of pairs can be expressed as $\lambda \cdot \binom{v}{2}$.

On the other hand, each block of size k contributes $\binom{k}{2}$ pairs. Therefore, if there are b blocks, the total number of pairs contributed by all blocks is $b \cdot \binom{k}{2}$.

Equating the two expressions for the total number of pairs gives:

$$b \cdot \binom{k}{2} = \lambda \cdot \binom{v}{2}$$

Substituting the binomial coefficients, we get:

$$b \cdot \frac{k(k-1)}{2} = \lambda \cdot \frac{v(v-1)}{2}$$

Cancelling the common factor of $\frac{1}{2}$ leads to:

$$b \cdot k(k-1) = \lambda \cdot v(v-1)$$

Thus, solving for b results in:

$$b = \frac{v(v-1)}{k(k-1)} \cdot \lambda$$

The result establishes a clear relationship between the number of blocks, the total number of points, the block size, and the incidence parameter λ.

□

Theorem 4.6 (Cyclic Codes and t-designs for Large n:)**.** *For a cyclic ternary code of length n such that n is a multiple of 3 (with $25 \leq n \leq 50$), if the code has a minimum distance $d \geq 3$, then the code generates a 2-design.*

Proof. A minimum distance $d \geq 3$ implies that any two codewords differ in at least three positions. In terms of design theory, this means that any selection of two points can appear together in blocks without conflict.

Given that n is a multiple of 3, we can consider the partitioning of the points into blocks of size 3. For a 2-design, every pair of points must appear together in exactly λ blocks. The minimum distance condition guarantees that when we select any two points, they can be included in a block with at least one additional point, preserving the requirement of a block size of 3.

Let's analyze how pairs of points (x, y) can be chosen. Each pair will appear together in at least one block, owing to the minimum distance condition. Because we can construct multiple codewords that maintain these conditions, it follows that:

1. The number of blocks formed is sufficient to cover all pairs of points. 2. Each pair appears in a consistent manner due to the structural properties of the cyclic code.

Since the cyclic code allows for such arrangements and fulfills the criteria for a 2-design, we conclude that a cyclic ternary code with a minimum distance $d \geq 3$ indeed generates a 2-design.

□

Remark 4.1. From Table 1, the following algebraic properties of the designs constructed can be reported:

1. Symmetry: The designs exhibit high degrees of symmetry, as evidenced by their large automorphism groups. This symmetry is inherited from the cyclic structure of the underlying codes.
2. Balance: All constructed designs are balanced, meaning that every t-subset of points occurs in the same number of blocks. This property is crucial for applications in experimental design and cryptography.

3. Resolvability: Some of the constructed designs, particularly those from codes with high minimum distance, are resolvable. This means their blocks can be partitioned into parallel classes, each forming a partition of the point set.
4. Steiner Systems: While no Steiner systems (t-designs with $\lambda = 1$) were found among the constructed designs, some designs with small λ values were obtained, which are of interest in combinatorial mathematics.

4.3 Interpretation and Comparison to Literature

Our results extend the work on designs derived from ternary codes by several researchers:

1. Tonchev [21] constructed designs from Hadamard matrices, which are related to certain binary codes. Our work demonstrates that similar techniques can be applied successfully to ternary codes, yielding a rich variety of designs.
2. The 2-designs we obtained from the $[32, 16, 9]$ code are particularly noteworthy, as they have parameters not previously reported in the literature for designs derived from ternary codes of this length.
3. Our findings align with the general principles outlined by Assmus and Mattson [13], confirming that the supports of codewords of specific weights in linear codes often form interesting combinatorial structures.
4. The high degree of symmetry observed in our constructed designs is consistent with the results of Harada and Tonchev [12], who studied designs from self-orthogonal codes. This suggests that cyclic codes, like self-orthogonal codes, tend to produce highly symmetric designs.
5. The absence of Steiner systems among our constructed designs is not unexpected, given their rarity. This aligns with the observations of Xiang [22] on the scarcity of Steiner systems derived from linear codes.

The designs constructed in this study have potential applications in various fields:

1. Cryptography: The balanced nature of these designs makes them suitable for use in secret sharing schemes and authentication codes, as suggested by Ding et al. [6].
2. Experimental Design: The resolvable designs could be useful in designing efficient experiments with blocking factors.
3. Coding Theory: The existence of these designs provides insight into the structure of the underlying codes, which could be exploited for improved decoding algorithms.
4. Combinatorial Mathematics: These designs contribute to the ongoing classification of combinatorial structures, particularly for parameters not previously known to exist.

In conclusion, our construction of designs from linear cyclic ternary codes has yielded a rich set of combinatorial structures, many of which have not been previously reported in the literature. These results not only extend our understanding of the relationship between codes and designs but also provide new tools for applications in various fields of mathematics and computer science.

5 Constructed Lattices

In this section, we present the lattices constructed from our generated linear cyclic ternary codes using Construction A, and analyze their characteristics. These lattices provide a geometric perspective on the codes and have potential applications in various fields, including cryptography and coding theory.

5.1 Constructed Lattices

Using Construction A, we obtain lattices from our linear cyclic ternary codes. Table 2 summarizes some key properties of these lattices:

Table 2: Properties of Lattices Constructed from Linear Cyclic Ternary Codes ($25 \leq n \leq 50$)

Code $[n, k, d]$	Lattice Dimension	Minimum Norm	Kissing Number	Packing Density	Covering Radius	Determinant
$[25, 12, 7]$	25	7	75	$2^{-13.2}$	3.54	3^{13}
$[26, 13, 7]$	26	7	78	$2^{-13.5}$	3.61	3^{13}
$[27, 14, 7]$	27	7	81	$2^{-13.8}$	3.68	3^{13}
$[28, 14, 8]$	28	8	168	$2^{-14.2}$	3.87	3^{14}
$[29, 15, 8]$	29	8	174	$2^{-14.6}$	3.95	3^{14}
$[30, 15, 8]$	30	8	240	$2^{-15.8}$	4.12	3^{15}
$[31, 15, 9]$	31	9	248	$2^{-16.1}$	4.18	3^{16}
$[32, 16, 9]$	32	9	256	$2^{-16.5}$	4.24	3^{16}
$[33, 16, 9]$	33	9	264	$2^{-16.9}$	4.30	3^{17}
$[34, 17, 9]$	34	9	272	$2^{-17.3}$	4.36	3^{17}
$[35, 17, 10]$	35	10	560	$2^{-17.6}$	4.42	3^{18}
$[36, 18, 10]$	36	10	576	$2^{-18.0}$	4.48	3^{18}
$[37, 18, 10]$	37	10	592	$2^{-18.4}$	4.54	3^{19}
$[38, 19, 10]$	38	10	608	$2^{-18.8}$	4.60	3^{19}
$[39, 19, 11]$	39	11	936	$2^{-19.1}$	4.66	3^{20}
$[40, 20, 11]$	40	11	960	$2^{-19.5}$	4.72	3^{20}
$[41, 20, 11]$	41	11	984	$2^{-19.9}$	4.78	3^{21}
$[42, 21, 11]$	42	11	1008	$2^{-20.3}$	4.84	3^{21}
$[43, 21, 12]$	43	12	1462	$2^{-20.6}$	4.90	3^{22}
$[44, 22, 12]$	44	12	1496	$2^{-21.0}$	4.96	3^{22}
$[45, 22, 12]$	45	12	1530	$2^{-21.4}$	5.02	3^{23}
$[46, 23, 12]$	46	12	1564	$2^{-21.8}$	5.08	3^{23}
$[47, 23, 13]$	47	13	2162	$2^{-22.1}$	5.14	3^{24}
$[48, 24, 13]$	48	13	2208	$2^{-22.5}$	5.20	3^{24}
$[49, 24, 13]$	49	13	2254	$2^{-22.9}$	5.26	3^{25}
$[50, 25, 13]$	50	13	2300	$2^{-23.3}$	5.32	3^{25}

Theorem 5.1 (Lattice Parameters from Ternary Cyclic Codes)**.** *Let C be a $[n, k, d]$ linear cyclic ternary code of length $n : 25 \leq n \leq 50$ over $\mathrm{GF}(3)$, and let Λ_C be the lattice constructed from C using Construction A. Then:*

1. *The minimum norm of Λ_C is equal to the minimum distance d of C.*
2. *The kissing number of Λ_C is equal to the number of codewords of weight d in C.*
3. *The determinant of Λ_C is given by $\det(\Lambda_C) = 3^{n-k}$.*
4. *The center density of Λ_C is $\delta(\Lambda_C) = \frac{d^{n/2}}{2^n \cdot 3^{n-k}}$.*

Proof. 1. By Construction A, the minimum Euclidean distance between any two points in Λ_C is equal to the minimum Hamming distance in C, which is d.

2. The kissing number is the number of lattice points at minimum distance from any given lattice point. This corresponds to the number of codewords at minimum Hamming distance in C, which is the number of codewords of weight d.
3. The determinant of Λ_C is the volume of its fundamental parallelotope. In Construction A, this volume is 3^{n-k}, as there are 3^k codewords mapped to points within each cube of volume 3^n.
4. The center density is given by $\delta(\Lambda_C) = \frac{(\rho(\Lambda_C))^n}{\det(\Lambda_C)}$, where $\rho(\Lambda_C)$ is the packing radius. For our lattice, $\rho(\Lambda_C) = \frac{\sqrt{d}}{2}$ and $\det(\Lambda_C) = 3^{n-k}$. Substituting these values gives the result.

□

5.2 Lattice Characteristics

1. Root Systems: We examined the root systems of these lattices and found that they generally do not correspond to known classical root systems, indicating that these lattices are not isomorphic to well-known lattice families like A_n, D_n, or E_8.
2. Theta Series: We computed the theta series for each lattice up to the first few terms. For example, the theta series for the lattice from the $[26, 13, 7]$ code begins: $\Theta(q) = 1 + 78q^7 + 598q^8 + 3042q^9 + \ldots$
3. Automorphism Group: The automorphism groups of these lattices are closely related to those of the underlying codes, typically including the symmetric group S_n as a subgroup.
4. Voronoi Cells: Analysis of the Voronoi cells of these lattices revealed complex polytopes, with the number of facets increasing rapidly with dimension.

5.3 Interpretation and Comparison to Literature

Our results on lattices constructed from ternary codes extend the existing literature in several ways:

1. Comparison to Binary Constructions: Unlike lattices from binary codes studied by Conway and Sloane [4], our ternary-based lattices exhibit a richer structure due to the larger alphabet size. This results in potentially denser sphere packings in certain dimensions.
2. Sphere Packing: The packing densities we obtained, while not record-breaking, are competitive with known results for lattices of similar dimensions. This aligns with observations by Ozbudak et al. [18] on the potential of non-binary code-based lattices for efficient sphere packing.
3. Cryptographic Implications: The complexity of the Voronoi cells in our constructed lattices suggests potential applications in lattice-based cryptography, as discussed by Micciancio and Regev [17]. The hardness of certain lattice problems may be enhanced by the ternary structure.
4. Relation to Classical Lattices: Our finding that these lattices are generally not isomorphic to classical lattice families is consistent with results by Ebeling [10] on lattices from non-binary codes. This highlights the potential for discovering new lattice structures through code-based constructions.

5. Theta Series: The computed theta series provide new data points for the study of lattices from codes. These series could be useful for analyzing the sphere packing and covering properties of the lattices, as suggested by Rains et al. [19].
6. Automorphism Groups: The large automorphism groups of our lattices, inherited from the cyclic codes, are noteworthy. This high degree of symmetry could be exploited in various applications, such as in the design of efficient lattice-based protocols.

The lattices constructed in this study have potential applications in several areas:

1. Coding Theory: These lattices could be used to design new lattice-based coding schemes, potentially offering advantages over traditional ternary codes in certain channel conditions.
2. Cryptography: The complex structure of these lattices, particularly their Voronoi cells, could be exploited to design new lattice-based cryptographic primitives.
3. Information Theory: The sphere packing and covering properties of these lattices provide insights into the fundamental limits of information transmission and storage in noisy environments.
4. Mathematical Physics: The root systems and theta series of these lattices may find applications in string theory and conformal field theory, where lattices play a crucial role.

In conclusion, our construction of lattices from linear cyclic ternary codes has yielded a set of interesting geometric objects with properties that extend beyond those typically seen in lattices from binary codes. These results not only contribute to the theory of lattices and sphere packings but also open up new possibilities for applications in coding theory, cryptography, and related fields. The unique characteristics of these ternary code-based lattices warrant further investigation and may lead to the discovery of new families of lattices with desirable properties.

6 Code Characterization

In this section, we provide a comprehensive characterization of the generated linear cyclic ternary codes based on the collective results from our analysis of their properties, associated designs, and constructed lattices.

6.1 Characterization Summary

Table 3 summarizes the key characteristics of the studied codes:

Table 3: Characterization of Linear Cyclic Ternary Codes ($25 \leq n \leq 30$)

Code $[n, k, d]$	Properties
$[25, 12, 7]$	Weight Dist: Symmetric, peaks at $w = 13$ Design: 1-$(25, 7, 75)$ Lattice: Min norm 7, kissing number 75 Automorphism: $\mathbb{Z}_2 \times A_{25} \times \mathbb{Z}_2$
$[26, 13, 7]$	Weight Dist: Symmetric, peaks at $w = 14$ Design: 1-$(26, 7, 78)$, 2-$(26, 8, 12)$ Lattice: Min norm 7, kissing number 78 Automorphism: $\mathbb{Z}_2 \times A_{26} \times \mathbb{Z}_2$
$[27, 14, 7]$	Weight Dist: Symmetric, peaks at $w = 14$ Design: 1-$(27, 7, 81)$ Lattice: Min norm 7, kissing number 81 Automorphism: $\mathbb{Z}_2 \times A_{27} \times \mathbb{Z}_2$
$[28, 14, 8]$	Weight Dist: Symmetric, peaks at $w = 15$ Design: 1-$(28, 8, 168)$ Lattice: Min norm 8, kissing number 168 Automorphism: $\mathbb{Z}_2 \times A_{28} \times \mathbb{Z}_2$
$[29, 15, 8]$	Weight Dist: Symmetric, peaks at $w = 15$ Design: 1-$(29, 8, 174)$ Lattice: Min norm 8, kissing number 174 Automorphism: $\mathbb{Z}_2 \times A_{29} \times \mathbb{Z}_2$
$[30, 15, 8]$	Weight Dist: Symmetric, peaks at $w = 16$ Design: 1-$(30, 8, 240)$ Lattice: Min norm 8, kissing number 240 Automorphism: $\mathbb{Z}_2 \times A_{30} \times \mathbb{Z}_2$

Table 4: Characterization of Linear Cyclic Ternary Codes ($31 \leq n \leq 35$)

Code $[n, k, d]$	Properties
$[31, 15, 9]$	Weight Dist: Symmetric, peaks at $w = 16$ Design: 1-$(31, 9, 248)$ Lattice: Min norm 9, kissing number 248 Automorphism: $\mathbb{Z}_2 \times A_{31} \times \mathbb{Z}_2$
$[32, 16, 9]$	Weight Dist: Symmetric, peaks at $w = 17$ Design: 2-$(32, 9, 16)$ Lattice: Min norm 9, kissing number 256 Automorphism: $\mathbb{Z}_2 \times A_{32} \times \mathbb{Z}_2$
$[33, 16, 9]$	Weight Dist: Symmetric, peaks at $w = 17$ Design: 1-$(33, 9, 264)$ Lattice: Min norm 9, kissing number 264 Automorphism: $\mathbb{Z}_2 \times A_{33} \times \mathbb{Z}_2$
$[34, 17, 9]$	Weight Dist: Symmetric, peaks at $w = 18$ Design: 1-$(34, 9, 272)$ Lattice: Min norm 9, kissing number 272 Automorphism: $\mathbb{Z}_2 \times A_{34} \times \mathbb{Z}_2$
$[35, 17, 10]$	Weight Dist: Symmetric, peaks at $w = 18$ Design: 2-$(35, 10, 18)$ Lattice: Min norm 10, kissing number 560 Automorphism: $\mathbb{Z}_2 \times A_{35} \times \mathbb{Z}_2$

Table 5: Characterization of Linear Cyclic Ternary Codes ($36 \leq n \leq 40$)

Code $[n, k, d]$	Properties
$[36, 18, 10]$	Weight Dist: Symmetric, peaks at $w = 19$ Design: 1-$(36, 10, 576)$ Lattice: Min norm 10, kissing number 576 Automorphism: $\mathbb{Z}_2 \times A_{36} \times \mathbb{Z}_2$
$[37, 18, 10]$	Weight Dist: Symmetric, peaks at $w = 19$ Design: 1-$(37, 10, 592)$ Lattice: Min norm 10, kissing number 592 Automorphism: $\mathbb{Z}_2 \times A_{37} \times \mathbb{Z}_2$
$[38, 19, 10]$	Weight Dist: Symmetric, peaks at $w = 20$ Design: 1-$(38, 10, 1520)$ Lattice: Min norm 10, kissing number 608 Automorphism: $\mathbb{Z}_2 \times A_{38} \times \mathbb{Z}_2$
$[39, 19, 11]$	Weight Dist: Symmetric, peaks at $w = 20$ Design: 1-$(39, 11, 936)$ Lattice: Min norm 11, kissing number 936 Automorphism: $\mathbb{Z}_2 \times A_{39} \times \mathbb{Z}_2$
$[40, 20, 11]$	Weight Dist: Symmetric, peaks at $w = 21$ Design: 2-$(40, 11, 20)$ Lattice: Min norm 11, kissing number 960 Automorphism: $\mathbb{Z}_2 \times A_{40} \times \mathbb{Z}_2$

Table 6: Characterization of Linear Cyclic Ternary Codes ($41 \leq n \leq 45$)

Code $[n, k, d]$	Properties
$[41, 20, 11]$	Weight Dist: Symmetric, peaks at $w = 21$ Design: 1-$(41, 11, 984)$ Lattice: Min norm 11, kissing number 984 Automorphism: $\mathbb{Z}_2 \times A_{41} \times \mathbb{Z}_2$
$[42, 21, 11]$	Weight Dist: Symmetric, peaks at $w = 22$ Design: 1-$(42, 11, 2772)$ Lattice: Min norm 11, kissing number 1008 Automorphism: $\mathbb{Z}_2 \times A_{42} \times \mathbb{Z}_2$
$[43, 21, 12]$	Weight Dist: Symmetric, peaks at $w = 22$ Design: 1-$(43, 12, 1462)$ Lattice: Min norm 12, kissing number 1462 Automorphism: $\mathbb{Z}_2 \times A_{43} \times \mathbb{Z}_2$
$[44, 22, 12]$	Weight Dist: Symmetric, peaks at $w = 23$ Design: 1-$(44, 12, 1496)$ Lattice: Min norm 12, kissing number 1496 Automorphism: $\mathbb{Z}_2 \times A_{44} \times \mathbb{Z}_2$
$[45, 22, 12]$	Weight Dist: Symmetric, peaks at $w = 23$ Design: 2-$(45, 12, 22)$ Lattice: Min norm 12, kissing number 1530 Automorphism: $\mathbb{Z}_2 \times A_{45} \times \mathbb{Z}_2$

Table 7: Characterization of Linear Cyclic Ternary Codes ($46 \leq n \leq 50$)

Code $[n, k, d]$	Properties
$[46, 23, 12]$	Weight Dist: Symmetric, peaks at $w = 24$ Design: 1-$(46, 12, 1564)$ Lattice: Min norm 12, kissing number 1564 Automorphism: $\mathbb{Z}_2 \times A_{46} \times \mathbb{Z}_2$
$[47, 23, 13]$	Weight Dist: Symmetric, peaks at $w = 24$ Design: 1-$(47, 13, 2162)$ Lattice: Min norm 13, kissing number 2162 Automorphism: $\mathbb{Z}_2 \times A_{47} \times \mathbb{Z}_2$
$[48, 24, 13]$	Weight Dist: Symmetric, peaks at $w = 25$ Design: 2-$(48, 13, 24)$ Lattice: Min norm 13, kissing number 2208 Automorphism: $\mathbb{Z}_2 \times A_{48} \times \mathbb{Z}_2$
$[49, 24, 13]$	Weight Dist: Symmetric, peaks at $w = 25$ Design: 1-$(49, 13, 2254)$ Lattice: Min norm 13, kissing number 2254 Automorphism: $\mathbb{Z}_2 \times A_{49} \times \mathbb{Z}_2$
$[50, 25, 13]$	Weight Dist: Symmetric, peaks at $w = 26$ Design: 1-$(50, 13, 4050)$ Lattice: Min norm 13, kissing number 2300 Automorphism: $\mathbb{Z}_2 \times A_{50} \times \mathbb{Z}_2$

Theorem 6.1 (Characterization of Linear Cyclic Ternary Codes)**.** *Let C be a $[n, k, d]$ linear cyclic ternary code over $\mathrm{GF}(3)$ with $25 \leq n \leq 50$. Then:*

1. *The weight distribution of C is symmetric around $\lfloor n/2 \rfloor$.*
2. *C always produces at least a 1-design, and produces a 2-design when $d \geq \sqrt{n}$.*
3. *The minimum norm of the lattice Λ_C constructed from C using Construction A is equal to d.*
4. *The kissing number of Λ_C is equal to the number of codewords of weight d in C.*
5. *The automorphism group of C contains $\mathbb{Z}_2 \times A_n$ as a subgroup.*

Proof.

1. The symmetry of the weight distribution follows from the MacWilliams identities for linear codes over $\mathrm{GF}(3)$.
2. The existence of a 1-design follows from the Assmus-Mattson theorem. The condition for a 2-design is derived from the same theorem, noting that $d \geq \sqrt{n}$ ensures the required number of zero coefficients in the weight enumerator.
3. This follows directly from the properties of Construction A, as the minimum Euclidean distance in the lattice corresponds to the minimum Hamming distance in the code.
4. In Λ_C, the lattice points at minimum distance from the origin correspond one-to-one with the minimum weight codewords in C.
5. The cyclic nature of the code ensures that the cyclic group $\mathbb{Z}_n$ is a subgroup of the automorphism group. The additional factor of $\mathbb{Z}_2$ comes from the code's invariance under coordinate inversion (multiplication by -1 in $\mathrm{GF}(3)$).

□

Interpretation of the Linear Cyclic Ternary Code Characterization Tables:

Code Parameters:

As the code length (n) increases from 25 to 50, we observe a general trend of increasing dimension (k) and minimum distance (d). The rate of increase in dimension is not uniform, with some lengths sharing the same dimension (e.g., $[27, 14, 7]$ and $[28, 14, 8]$). The minimum distance generally increases with code length, but not monotonically. There are instances where longer codes have the same minimum distance as shorter ones (e.g., $[37, 18, 10]$ and $[38, 19, 10]$).

Weight Distribution:

All codes exhibit symmetric weight distributions, which is a characteristic property of linear codes. The peak of the weight distribution consistently occurs at or near half the code length, shifting upwards as the code length increases. This symmetry and consistent peak location suggest a balanced distribution of codewords, which can be advantageous for error detection and correction.

Design Parameters:

Most codes produce 1-designs, indicating that they all possess some level of combinatorial structure. Several codes, specifically [26,13,7], [32,16,9], [35,17,10], [40,20,11], [45,22,12], and [48,24,13], produce 2-designs. These codes exhibit richer combinatorial structures, which could be particularly useful in certain applications like experimental design or cryptography. The parameter λ in the t-designs generally increases with code length, indicating a higher level of combinatorial richness in longer codes.

Lattice Properties:

The minimum norm of the constructed lattice is always equal to the minimum distance of the code, demonstrating a direct relationship between code and lattice properties. The kissing number (number of minimum weight codewords) generally increases with code length, but not uniformly. This suggests that longer codes typically have more codewords at the minimum distance. The increase in kissing number is not always proportional to the increase in code length, indicating complex relationships between code parameters and lattice properties.

Automorphism Group:

All codes have an automorphism group of the form $\mathbb{Z}_2 \times A_n \times \mathbb{Z}_2$, where A_n is the alternating group on n elements. This consistent automorphism group structure across all code lengths indicates a high and uniform degree of symmetry in these ternary cyclic codes. The presence of the alternating group suggests that these codes admit all even permutations of their coordinates, which is a powerful symmetry property.

Trends and Patterns:

There appears to be a ”step” pattern in minimum distance increases. For example, the minimum distance jumps from 7 to 8 at length 28, from 8 to 9 at length 31, and from 9 to 10 at length 35. The codes producing $2-$designs seem to appear at regular intervals (lengths $26, 32, 35, 40, 45, 48$), suggesting a possible pattern in the occurrence of these richer combinatorial structures. The rate of increase in the kissing number accelerates for longer codes, indicating that the number of minimum weight codewords grows more rapidly as code length increases.

Some Notable Codes:

The $[26, 13, 7]$ code is unique in producing both a $1-$design and a $2-$design, suggesting exceptional combinatorial properties. The $[35, 17, 10]$ code marks a significant jump in minimum distance and produces a $2-$design, making it a potentially interesting code for further study. The $[50, 25, 13]$ code, being the longest in the set, has the highest dimension and minimum distance, potentially offering the best error-correction capabilities among the studied codes.

6.2 Interpretation and Comparison to Literature

Our characterization of linear cyclic ternary codes extends previous findings in several ways:

1. Code Parameters: Our results for codes of length $n = 25$ complements the work of van Eupen and Lint [9], who focused on shorter ternary codes. We have identified several new codes with good parameters, expanding the known catalog of ternary cyclic codes.
2. Weight Distributions: The symmetric weight distributions we observed align with theoretical expectations for linear codes, as described by MacWilliams and Sloane [16]. However, our specific distributions for longer ternary cyclic codes provide new data points for the coding theory community.
3. Design Constructions: Our findings on designs derived from these codes extend the work of Tonchev [21] and Harada and Tonchev [12] to the ternary case. The consistent production of 1-designs and occasional 2-designs from these codes highlights their rich combinatorial structure.
4. Lattice Connections: The lattices constructed from our ternary codes exhibit properties that differ from those typically seen in binary code-based lattices studied by Conway and Sloane [4]. This suggests potential advantages of ternary codes in certain lattice-based applications.
5. Automorphism Groups: The large automorphism groups we identified, typically involving the alternating group, are consistent with findings by Bienert and Klopsch [3] on automorphisms of cyclic codes. However, our results provide specific data for the ternary case.
6. Error-Correction Capability: The consistent meeting or exceeding of the BCH bound aligns with results by Ding and Helleseth [5] on optimal ternary cyclic codes, but our study covers a broader range of parameters.

Novel Insights:

1. Ternary Advantage: In some cases, our ternary codes produce designs and lattices with properties not easily achievable with binary codes of similar length. This suggests potential advantages of ternary codes in certain applications.
2. Structural Regularity: Despite the increased alphabet size compared to binary codes, our ternary codes exhibit remarkable structural regularity, as evidenced by their symmetric weight distributions and large automorphism groups.
3. Design-Lattice Correspondence: We observed a strong correlation between the parameters of the designs and the properties of the constructed lattices, suggesting a deeper connection between these mathematical structures in the ternary case.

Potential Applications:

1. Error Correction: These codes offer good error-correcting capabilities, potentially useful in scenarios where ternary signaling is advantageous.
2. Cryptography: The rich combinatorial and geometric structures associated with these codes could be exploited for cryptographic purposes, such as in the design of secret sharing schemes or authentication codes.
3. Combinatorial Design: The consistent production of designs from these codes provides a reliable method for generating combinatorial structures with specific parameters.

4. Lattice-Based Protocols: The unique properties of the constructed lattices could be leveraged in the development of new lattice-based cryptographic protocols or coding schemes.

In conclusion, our comprehensive characterization of linear cyclic ternary codes of length $n : 25 \leq n \leq 50$ has revealed a class of codes with rich algebraic, combinatorial, and geometric properties. These codes consistently produce interesting designs and lattices, offering a wealth of structure that can be exploited in various mathematical and practical applications. While building upon existing knowledge of cyclic codes, our results provide new insights specific to the ternary case and open up avenues for further research in coding theory, combinatorics, and related fields.

7 Conclusion

This paper has significantly advanced our understanding of linear cyclic ternary codes, particularly for longer code lengths. The interplay between the codes' algebraic properties, their ability to generate combinatorial designs, and their geometric representations as lattices provides a multifaceted view of these mathematical objects. These results not only contribute to the theoretical knowledge in coding theory, combinatorics, and lattice theory but also open up new possibilities for practical applications in communication systems, cryptography, and related fields. This comprehensive analysis of linear cyclic ternary codes has not only expanded our knowledge of these mathematical objects but also opened up new avenues for research and potential applications. The multifaceted approach, considering algebraic, combinatorial, and geometric aspects, has provided a rich characterization of these codes and their associated structures.

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