Derivation and solution of the heat equation in 1-D

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Abstract

Heat flows in the direction of decreasing temperature, that is, from hot to cool. In this paper we derive the heat equation and consider the flow of heat along a metal rod. The rod allows us to consider the temperature, $u(x,t)$, as one dimensional in x but changing in time, t.

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1 Introduction

The heat equation is an important partial differential equation (PDE) which describes the distribution of heat (or variation in temperature) in a given region over time. Heat is a process of energy transfer as a result of temperature difference between the two points. Thus, the term 'heat' is used to describe the energy transferred through the heating process. Temperature, on the other hand, is a physical property of matter that describes the hotness or coldness of an object or environment. Therefore, no heat would be exchanged between bodies of the same temperature, Christopher Yaluma [6]. In an object, heat will flow in the direction of decreasing temperature. The heat flow is proportional to the temperature gradient, that is;

$$-k\frac{\partial u}{\partial x},$$

where k is a constant of proportionality. Consider a small element of the rod between the positions x and $x + \delta x$. The amount of heat in the element, at time t, is

$$H(t) = \sigma\rho u(x,t)\delta x,$$

where σ is the specific heat of the rod and ρ is the mass per unit length. At time $t + \delta t$, the amount of heat is

$$H(t+\delta t) = \sigma\rho u(x, t+\delta t)\delta x$$

Thus, the change in heat is simply

$$H(t+\delta t) - H(t) = \sigma\rho\left(u(x,t+\delta t) - u(x,t)\right)\delta x.$$

This change of heat must equal the heat flowing in at x minus the heat flowing out at $x + \delta x$ during the time interval δt. This may be expressed as

$$\left[\left(-k\frac{\partial u}{\partial x}\right)_x - \left(-k\frac{\partial u}{\partial x}\right)_{x+\delta x}\right]\delta t$$

Equating these expressions and dividing by δx and δt gives,

$$\sigma\rho\frac{u(x,t+\delta t) - u(x,t)}{\delta t} = k\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)}{\delta x}.$$

Taking the limits of δx and δt tending to zero, we obtain the partial derivatives, John Fritz *et al* [3]. Hence, the heat equation in 1-D is

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$$

where $c^2 = k/\sigma\rho$ is the constant thermal conductivity and $\partial^2 u/\partial x^2$ is the thermal conduction.
This is in the form, Evans *et al* [2];

$$u_t = c^2 u_{xx} \tag{1}$$

The heat equation has the same form as the equation describing diffusion, Thambynayagam [4].
By separation of variables;
Let

$$U = XT, X = X(x), T = T(t) \tag{2}$$

Equation (1) now becomes;

$$XT^{'} = c^2 X^{''} T \tag{3}$$

Separating the variables;

$$\frac{1}{c^2}\frac{T'}{T} = \frac{X''}{X} = k \tag{4}$$

Thus;

$$X'' - kX = 0 \tag{5}$$

$$T' - c^2 kT = 0 \tag{6}$$

(i) For k=0; $X'' = 0 \Rightarrow X' = C_1 \Rightarrow X = C_1 x + C_2$ and $T' = 0 \Rightarrow T = C_3$

$$\Rightarrow u = XT = (C_1 x + C_2)C_3 = C_1 C_3 x + C_2 C_3 \tag{7}$$

(ii) For $k > 0$, say $k = \rho^2$, $X'' - \rho^2 X$=0 and $\frac{T'}{T} = c^2\rho^2$

$$\Rightarrow r = \pm\rho \Rightarrow X = C_1 e^{\rho x} + C_2 e^{-\rho x}$$

$$\Rightarrow lnT = c^2\rho^2 t + C_3 \Rightarrow T = e^{c^2\rho^2 t + C_3} \Rightarrow T = C_3 e^{c^2\rho^2 t}$$

$$\Rightarrow U = XT = (C_1 e^{\rho x} + C_2 e^{-\rho x})C_3 e^{c^2\rho^2 t} \tag{8}$$

(iii) For $k < 0$, say $k = -\rho^2, X'' + \rho^2 X = 0 \Rightarrow r = \pm i\rho$

$$\Rightarrow \frac{T'}{T} = -c^2\rho^2 \Rightarrow lnT = -c^2\rho^2 t + C_3$$

$$\Rightarrow T = e^{-c^2\rho^2 t + C_3} \Rightarrow T = C_3 e^{-c^2\rho^2 t}$$

$$\Rightarrow U = XT = (C_1 \cos\rho x + C_2 \sin\rho x)C_3 e^{-c^2\rho^2 t}$$

$$\Rightarrow U(x,t) = (A\cos\rho x + B\sin\rho x)e^{-c^2\rho^2 t} \tag{9}$$

This is consistent with the physical nature of the periodic equation.

2 Heat flow in a metal rod

We consider a metal rod with boundary conditions (BC), Carslaw *et al* [1]

$$x = 0; U(0,t) = 0; x = l; U(l,t) = 0; \forall t$$

With Initial conditions (IC);

$$t = 0; U(x,0) = u_0$$

With

$$x = 0; U(0,t) = Ae^{-c^2\rho^2 t} = 0, \Rightarrow A = 0$$
$$\Rightarrow U(x,t) = B\sin\rho x e^{-c^2\rho^2 t}$$

With

$$x = l, U(l,t) = B\sin\rho l e^{-c^2\rho^2 t} = 0$$
$$\Rightarrow \sin\rho l = 0 \Rightarrow \rho l = n\pi \Rightarrow \rho = \frac{n\pi}{l}$$

Thus

$$U_n(x,t) = B_n \sin\frac{n\pi x}{l} e^{-c^2(\frac{n\pi}{l})^2 t}$$

This can be generalized as

$$U_n(x,t) = b_n \sin\frac{n\pi x}{l} e^{-c^2(\frac{n\pi}{l})^2 t}$$

with

$$b_n = B_n$$

Thus the general solution is

$$U(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} e^{-c^2(\frac{n\pi}{l})^2 t} \tag{10}$$

From the Initial condition, we have

$$U(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} = u_0$$

So that

$$u_0 = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \tag{11}$$

This is just the half range sine series, Weisstein *et al* [5] where;

$$b_n = \frac{2}{l}\int_0^1 f(x)\sin\frac{n\pi x}{l}\,dx \tag{12}$$

for all positive integers, n

3 Conclusion

It is worth noting that because every term in the solution for $U(x,t)$ has a negative exponential in it, the temperature must decrease in time and the final solution will tend to $U = 0$. This is different from the wave equation where the oscillations simply continued for all time. This trivial solution, $U = 0$, is a consequence of the particular boundary conditions chosen here.

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