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Equivalent Banach Operator Ideal Norms[1]

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Abstract

Let X, Y be Banach spaces and consider the w'-topology (the dual weak operator topology) on the space $(L(X,Y), \|.\|)$ of bounded linear operators from X into X with the uniform operator norm. $L^{w'}(X,Y)$ is the space of all $T \in L(X,Y)$ for which there exists a sequence of compact linear operators $(Tn) \subset K(X,Y)$ such that $T = w' - lim_n T_n$.

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Two equivalent norms,

$$|||T||| := inf\{ \sup_{n} \|T_n\| : T_n \in K(X,Y), T_n \overset{w'}{\to} T\} \; and$$

$$\|T\|_u := inf\{ \sup_{n} \{max\{\|T_n\|, \|T-2T_n\|\}\} : \| : T_n \in K(X,Y), T_n \overset{w'}{\to} T\}$$

on $L^{w'}(X,Y)$, are considered. We show that $(L^{w'}, |||.|||)$ and $(L^{w'}, \|.\|_u)$ are Banach operator ideals.

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1. Introduction

Throughout this paper X and Y will be Banach spaces. The space of bounded linear operators from X to Y is denoted by $L(X,Y)$ and the subspaces consisting of all finite rank bounded linear operators, all compact linear operators and all weakly compact linear operators, are denoted by $F(X,Y)$, $K(X,Y)$ and $W(X,Y)$, respectively. The closed unit ball of a Banach space X is denoted by B_X and the continuous dual space of X is denoted by X^*.

We follow the authors of the paper [3], calling a subspace X of a Banach space Y an *ideal* in Y if the annihilator $X^\perp$ of X is the kernel of a contractive projection P on the (continuous) dual space Y^* of Y, whose range is isomorphic to X^*. Since, by Hahn Banach Theorem such a projection has norm 1, it follows that $\|id_{Y^*} - 2P\| \geq 1$. Moreover, if the projection P exists on Y^* such that $ker\,P = X^\perp$ and $\|id_{Y^*} - 2P\| \leq 1$ (i.e $\|id_{Y^*} - 2P\| = 1$ in this case), then X is called a *u-ideal* (or *unconditional ideal*) in Y. This concept was introduced by Casazza and Kalton (cf [2]) and the equality $\|id_{Y^*} - 2P\| = 1$ is equivalent to requiring that if $\xi \in X^\perp, \phi \in V := P(Y^*)$, then $\|\phi + \xi\| = \|\phi - \xi\|$. The natural examples of u-ideals (with respect to their biduals) are order continuous Banach lattices-although there are many examples of u-ideals which are not Banach lattices. In a subsequent paper, the authors of [3] further investigated u-ideals along with so called h-ideals, in which case it is required that $\|\phi + \xi\| = \|\phi + \lambda\xi\|$ for all $\xi \in X^{\perp}$, $\phi \in V$ and all $|\lambda| = 1$. Much of the paper [3] is devoted to a general study of u-ideal and h-ideals. However, in section 8 of that paper, the authors find necessary conditions on a Banach space X such that the space $K(X)$ of compact operators is a u-ideal in the space $L(X)$ of bounded linear operators, showing that this is the case if X is separable and has (UKAP) (unconditional compact approximation property, i.e. if there exists a sequence (K_n) in $K(X)$ such that $\underset{n}{lim}\ K_n x = x$ for all $x \in X$ and $\underset{n}{lim}\ \|id_X - 2K_n\| = 1$).

Johnson proved in [7] that if Y is a Banach space having the bounded approximation property then the annihilator $K(X,Y)^{\perp}$ in the (continuous) dual space $L(X,Y)^*$ is the kernel of a projection on $L(X,Y)^*$. The range space of the projection is isomorphic to the dual space $K(X,Y)^*$. K. John showed in [5] that Johnson's result is also true in case of any separable Pisier space $X = P$ and its dual $Y = P^*$, both being spaces which do not have the approximation property. This motivated his more general results in a later paper,(cf [6]).

Following Kalton [8] we denote by w' the *dual weak operator topology* on $L(X,Y)$ which is defined by the linear functionals

$$T \mapsto e^{**}(T^* f^*), \quad f^* \in Y^*, e^{**} \in X^{**}.$$

Although the weak topology of $L(X,Y)$ is in general stronger than w', it is shown by Kalton in [8] that w'-compact subsets of $K(X,Y)$ are weakly compact. In particular,

- *If $(T_n) \subset K(X,Y)$ is a w'-convergent sequence which converges to a $T \in K(X,Y)$, then $T_n \to T$ in the weak topology of $L(X,Y)$.*

This result was used by K. John (in [6]) to show that if for each $T \in L(X,Y)$ there exists a sequence $T_n \subset K(X,Y)$ such that $T_n \to T$ in the dual weak operator topology, then the annihilator $K(X,Y)^{\perp}$ in $L(X,Y)^*$ is the kernel of a projection on $L(X,Y)^*$. In the paper [1] an alternative (operator ideal) approach is followed to prove similar (and more general) versions of John's results. In this paper we build on the results in [1] to show that $(L^{w'}, |||.|||)$ and $(L^{w'}, \|.\|_u)$ are Banach operator ideals.

2. Operator ideal properties.

Definition 2.1: *Let $T \in L(X,Y)$. T is said to have the w'-compact approximation property (w'-cap) if there is a sequence $(T_n) \subset K(X,Y)$ such that $T_n \overset{w'}{\to} T\}$. Let $L^{w'}(X,Y)$ be the family of all $T \in L(X,Y)$ which have the w'-compact approximation property.*

An easy application of the Uniform Boundedness Theorem shows that

Lemma 2.2: *If $T_n \to T$ in the w'-topology of $L(X,Y)$ then (T_n) is norm bounded.*

Let X, Y be fixed Banach spaces. For $T \in L^{w'}(X,Y)$ we put

$$(*) \qquad |||T||| := inf\{ \underset{n}{sup} \|T_n\| : T_n \in K(X,Y), T_n \overset{w'}{\to} T\}.$$

Clearly, if $T \in K(X,Y)$, then $|||T||| = \|T\|$.
Refer to [9] and [4] for information in connection with operator ideals. In particular we recall the following criteria for a subclass of the operator ideal $(L, \|.\|)$ to be a complete operator ideal on the family of all Banach spaces.

Theorem 2.3: *(cf. [9], 6.2.3, pp.91) Let U be a subclass of L with an $\Re^+$-valued function α such that the following conditions are satisfied:*

(i) If X, Y are Banach spaces, then $a \otimes y \in U(X,Y)$ for all $a \in X^*$, $y \in Y$ and $\alpha(a \otimes y) = \|a\|\|y\|$.

(ii) $RST \in U(X,Y)$ and $\alpha(RST) \leq \|R\|\alpha(S)\|T\|$ whenever $T \in L(X, X_0)$, $S \in U(X_0, Y_0)$ and $R \in L(Y_0, Y)$.

(iii) If $S_1, S_2, ... \in U(X,Y)$ and $\sum_{i=1}^{\infty} \alpha(S_i) < \infty$, then $S = \sum_{i=1}^{\infty} S_i = \|.\| - lim_n \sum_{i=1}^{\infty} S_i \in U(X,Y)$. And $\alpha(\sum_{i=1}^{\infty} S_i) \leq \sum_{i=1}^{\infty} \alpha(S_i)$.

Then (U, α) is a complete normed operator ideal.

This important result is instrumental in proving that $(L^{w'}, |||.|||)$ is a Banach operator ideal. This fact is proved in [2]. Both for the sake of completeness and later reference, we discuss the proof here.

Theorem 2.4: ([1], Theorem 2.4) *Let $L^{w'}$ denote the assignment which associates with each pair of Banach spaces X, Y the vector space $L^{w'}(X,Y)$. And let $|||.|||$ be the assignment that associates with every pair of Banach spaces X, Y and with every operator S belonging to $L^{w'}(X,Y)$ the real number $|||S|||$ in $(*)$. Then $(L^{w'}, |||.|||)$ is a Banach operator ideal.*

Proof: Notice that $\|.\| \leq |||.|||$ on $L^{w'}(X,Y)$, where $\|.\|$ is the uniform operator norm on $L(X,Y)$. In fact for any $\epsilon > 0$, let $\|x\| \leq 1$, $\|y^*\| \leq 1$ such that $\|T\| - \epsilon \leq |y^*(Tx)| = lim_n |y^*(T_n x)| \leq sup_n \|T_n\|$ where $(T_n) \subset K(X,Y)$ such that $T_n \overset{w'}{\to} T$. Clearly $\|T\| \leq |||T||| + \epsilon$. To prove that $(L^{w'}, |||.|||)$ is a complete normed ideal we make use of Theorem 2.3:

(i) $|||I_K||| = 1$ where $I_K \in L^{w'}(K)$ is the identity map on the 1-dimensional Banach space K.

(ii) Let $T \in L(X, X_0)$, $S \in L^{w'}(X, Y_0)$ *and* $R \in L(Y_0, Y)$. Then if $S_n \overset{w'}{\to} S$, $S_n \in K(X,Y)$ arbitrary, then $RS_nT \overset{w'}{\to} RST$. Hence

$$|||RST||| \leq \sup_n \|RS_nT\| \leq \|R\|(\sup_n \|S_n\|)\|T\|.$$

Since (S_n) was arbitrary chosen, it is clear that $|||RST||| \leq \|R\| \, |||S||| \, \|T\|$.

(iii) Now suppose that $(T_n) \subset L^{w'}(X,Y)$ with $\sum_{i=1}^{\infty} |||T_n||| < \infty$. We have to show that $\sum_{i=1}^{\infty} T_i = \|.\| - lim_n \sum_{i=1}^{\infty} T_i$ exists and is in $L^{w'}(X,Y)$ with

$|||\sum_{i=1}^{\infty} T_i||| \leq \sum_{i=1}^{\infty} |||T_i|||$: Let $T_{n,i} \in K(X,Y)$ such that $T_{n,i} \underset{n}{\overset{w'}{\to}} T_i$, $sup_n\|T_{n,i}\| \leq |||T_i||| + \frac{\epsilon}{2^i}$. For arbitrary $\|x^{**}\| \leq 1$, $\|y^*\| \leq 1$ we have $|x^{**}(T_{n,i}^* y^*)| \leq |||T_i||| + \frac{\epsilon}{2^i}$, $\forall\, i\ and \forall\, n$.

Hence $\sum_{i=1}^{\infty} x^{**}(T_{n,i}^* y^*)$ converges uniformly in $n \in \mathbf{N}$, thus showing that

$$(*) \qquad \sum_{i=1}^{\infty} x^{**}(T_i^* y^*) = \underset{n}{lim} \sum_{i=1}^{\infty} x^{**}(T_i^* y^*).$$

It follows from the completeness of $(L(X,Y), \|.\|)$ and $(K(X,Y), \|.\|)$ and the inequalities $\|T_i\| \leq |||T_i|||$ for all i and $\|T_{n,i}\| \leq |||T_i||| + \frac{\epsilon}{2^i}$ for all i, that $\sum_{i=1}^{\infty} T_i \in L(X,Y)$ and $\sum_{i=1}^{\infty} T_{n,i} \in K(X,Y)$ for all n. Since $(*)$ holds for arbitrary $x^{**} \in B_{X^{**}}$ and $y^* \in B_{Y^*}$, it follows that $\sum_{i=1}^{\infty} T_{n,i} \overset{w'}{\to} \sum_{i=1}^{\infty} T_i$. Hence $\sum_{i=1}^{\infty} T_i$ is in $L^{w'}(X,Y)$ and

$$|||\sum_{i=1} T_i||| \leq \underset{n}{sup} \|\sum_{i=1}^{\infty} T_{n,i}\| \leq \underset{n}{sup} \sum_{i=1}^{\infty} \|T_{n,i}\| \leq \epsilon + \sum_{i=1} |||T_i|||.$$

This shows that $\sum_{i=1}^{\infty} |||T_i||| \leq \sum_{i=1}^{\infty} |||T_i|||$. By theorem 2.3, $(L^{w'}, |||.|||)$ is a Banach ideal of operators.

Definition 2.5: *Let $T \in L^{w'}(X,Y)$ and suppose $(T_n) \subset K(X,Y)$ converges in the dual weak operator topology of T. We denote by $K_u((T_n))$ the number given by*

$$K_u((T_n)) := \underset{n \in \boldsymbol{N}}{sup} \{max\{\|T_n\|, \|T - 2T_n\|\}\},$$

which is a finite number because of the Uniform Boundeddness Theorem. The u-norm on $L^{w'}(X,Y)$ is then given by

$$\|T\|_u := inf\{K_u((T_n)) :\ T =\ w' - \underset{n}{lim}\ T_n,\ T_n \in K(X,Y)\}.$$

It is clear from the definition that $|||T||| \leq \|T\|_u$ for all $T \in L^{w'}(X,Y)$. Also, if $T \in K(X,Y)$ then we may put $T_n = T$ for all n, in which case $K_u((T_n)) = \|T\|$, showing that $\|T\|_u \leq \|T\|$; therefore we have $|||T||| = \|T\|_u = \|T\|$ for all $T \in K(X,Y)$.

Theorem 2.6: $(L^{w'}, \|.\|_u)$ *is a Banach operator ideal.*

Proof: (i) It is clear that $\|.\| \leq |||.||| \leq \|.\|_u$ *on* $L^{w'}(X,Y)$ for all Banach spaces X, Y and that the identity map on any 1-dimensional Banach space has u-norm 1.

(ii) For $T \in L(X, X_0)$, $R \in L(Y_0, Y)$, $S \in L^{w'}(X_0, Y_0)$ *and* $(S_n) \subset K(X_0, Y_0)$ such that $S_n \overset{w'}{\rightarrow} S$, we have

$$\|RST\|_u \leq K_u((RS_nT)) \leq \|R\|\|T\| \sup_n \{max\{\|S_n\|, \|S - 2S_n\|\}\} \leq \|R\|\|T\|K_u((S_n)).$$

The sequence $S_n \subset K(X_0, Y_0)$ being arbitrarily chosen to satisfy $S_n \overset{w'}{\rightarrow} S$, it follows that

$$\|RST\|_u \leq \|R\|\|S\|_u\|T\|.$$

(iii) Now suppose that $(T_n) \subset L^{w'}(X,Y)$ with $\sum_{i=1}^{\infty} \|T_n\|_u < \infty$. Since this implies that $\sum_{i=1}^{\infty} |||T_n|||_u < \infty$ and $(L^{w'}, |||.|||)$ is a Banach operator ideal, it follows that $\sum_{i=1}^{\infty} T_n \in L^{w'}(X,Y)$. We still have to prove that

$$\|\sum_{i=1}^{\infty} T_n\|_u \leq \sum_{i=1}^{\infty} \|T_n\|_u.$$

To do so, we choose for arbitrary $\epsilon > 0$ and each fixed $i \in \mathbf{N}$, a sequence $(T_{n,i}) \subset K(X,Y)$ such that $T_{n,i} \overset{w'}{\rightarrow} T_i$ if $n \rightarrow \infty$ and

$$K_u((T_{n,i})) \leq \|T_i\|_u + \frac{\epsilon}{2^i}.$$

As in the proof of Theorem 2.4 it follows that

$$\sum_{i=1}^{\infty} T_i = w' - \lim_n \sum_{i=1}^{\infty} T_{n,i} \in L^{w'}(X,Y).$$

Therefore, we have

$$\|\sum_{i=1} T_i\|_u \leq K_u((\sum_{i=1}^{\infty} T_{n,i})_n)$$

$$= \begin{matrix} sup \\ n \end{matrix} \left\{ max \left\{ \|\sum_{i=1}^{\infty} T_{n,i}\|, \|\sum_{i=1} T_i - 2\sum_{i=1}^{\infty} T_{n,i}\| \right\} \right\}$$

$$\leq \begin{matrix} sup \\ n \end{matrix} \left\{ max \left\{ \sum_{i=1}^{\infty} \|T_{n,i}\|, \sum_{i=1} \|T_i - 2\sum_{i=1}^{\infty} T_{n,i}\| \right\} \right\}$$

$$\leq \sum_{i=1} K_u((T_{n,i})) \leq \sum_{i=1} \|T_i\|_u + \epsilon,$$

which proves that

$$\|\sum_{i=1} T_i\|_u \leq \sum_{i=1} \|T_i\|_u.$$

We conclude that $(L^{w'}, \|.\|_u)$ is a Banach operator ideal.

Corollary 2.7: *The norms $|||.|||$ and $\|.\|_u$ are equivalent on $L^{w'}(X, Y)$ for all Banach spaces X, Y.*

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