

**SPACES OF COMPACT OPERATORS AND THEIR DUAL SPACES**

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The  $w'$ -topology on the space  $L(X, Y)$  of bounded linear operators from the Banach space  $X$  into the Banach space  $Y$  is discussed in [10]. Let  $\mathcal{L}^{w'}(X, Y)$  denote the space of all  $T \in L(X, Y)$  for which there exists a sequence of compact linear operators  $(T_n) \subset K(X, Y)$  such that  $T = w' - \lim_n T_n$  and let  $|||T||| := \inf \{ \sup_n \|T_n\| : T_n \in K(X, Y), T_n \xrightarrow{w'} T \}$ . We show that  $(\mathcal{L}^{w'}, ||| \cdot |||)$  is a Banach ideal of operators and that the continuous dual space  $K(X, Y)^*$  is complemented in  $(\mathcal{L}^{w'}(X, Y), ||| \cdot |||)^*$ . This results in necessary and sufficient conditions for  $K(X, Y)$  to be reflexive, whereby the spaces  $X$  and  $Y$  need not satisfy the approximation property. Similar results follow when  $X$  and  $Y$  are locally convex spaces.

**1. Introduction and Notation.**

The question of the complementation of the dual space  $K(X, Y)^*$  of the space of compact operators  $K(X, Y)$  in the dual space  $L(X, Y)^*$  of  $L(X, Y)$  (the space of bounded linear operators from  $X$  into  $Y$ ) when  $X$  and  $Y$  are Banach spaces, was more or less settled by a result of J. Johnson (in 1979). Johnson proved in [9] that if  $Y$  is a Banach space having the bounded approximation property then the annihilator  $K(X, Y)^\perp$  in the (continuous) dual space  $L(X, Y)^*$  is the kernel of a projection on  $L(X, Y)^*$ . The range space of the projection is isomorphic to the dual space  $K(X, Y)^*$ . Although many examples of Banach spaces which fail the approximation property became known after Enflo's example in 1973, the fact that the classical spaces have

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the approximation property made Johnson's result a "final" one. However, after Pisier's paper in 1983 (cf. [14]) which introduced counterexamples to a conjecture of Grothendieck in connection with the projective and injective norms on tensor products of Banach spaces, a new interest in the study of operators on Banach spaces without the approximation property was evoked. K. John's observation in [7] that Johnson's result is also true in case of any separable Pisier space  $X = \mathbf{P}$  and its dual  $Y = \mathbf{P}^*$  - they are both spaces which do not have the approximation property - motivated his more general results in [8] where it is for instance proved that Johnson's result holds for couples of Banach spaces  $X, Y$  such that each  $T : X \rightarrow Y$  factors through a Banach space  $Z$ , its dual  $Z^*$  being a separable Banach space which has the bounded approximation property. Moreover, a substantially generalised version of the result is proved there.

Following Kalton [10] we denote by  $w'$  the *dual weak operator topology* on  $L(X, Y)$  which is defined by the linear functionals

$$T \mapsto e^{**}(T^* f^*), \quad f^* \in Y^*, e^{**} \in X^{**}.$$

Although the weak topology of  $L(X, Y)$  is in general stronger than  $w'$ , it is shown by Kalton in [10] that  $w'$ -compact subsets of  $K(X, Y)$  are weakly compact. In particular, it is important to notice that

**THEOREM (K).** *If  $(T_n) \subset K(X, Y)$  is a  $w'$ -convergent sequence which converges to a  $T \in K(X, Y)$ , then  $T_n \rightarrow T$  in the weak topology of  $L(X, Y)$ .*

Using Theorem (K) and some fundamental properties of complete normed spaces, K. John proved (in [8]) that if for each  $T \in L(X, Y)$  there exists a sequence  $(T_n) \subset K(X, Y)$  such that  $T_n \rightarrow T$  in the dual weak operator topology, then the annihilator  $K(X, Y)^\perp$  in  $L(X, Y)^*$  is the kernel of a projection on  $L(X, Y)^*$ . The conditions of this theorem are for instance satisfied if each  $T \in L(X, Y)$  factors through a Banach space (depending on  $T$ ) which has a separable dual with the approximation property. For example, each  $T \in L(\mathbf{P}, \mathbf{P}^*)$  factors through a Hilbert space. In Section 1 we follow an operator ideal approach to John's results; in this way we believe the results are stated in a setting which allows us to prove extended versions of the same and other results. For instance, extended versions of results by Kalton and J. Johnson about reflexivity of the space  $K(X, Y)$  of compact operators are proved by means of our approach.

In the paper [4], the author considers generalisations of John's results in the setting of locally convex spaces. In order to do so, the author in [4] has to

consider a generalisation of Kalton's result Theorem (K) in connection with the weak convergence of  $w'$ -convergent sequences of compact operators to the setting of operators on locally convex spaces (cf. [4], Proposition 1). Then the proof of the final result about the existence of projections on dual spaces (cf. [4], Proposition 2) depends on complicated arguments which involve the application of the Riesz representation theorem (for  $C(K)$ -spaces) and the Lebesgue Dominated Convergence Theorem. In Section 2 we follow an alternative approach which results in both extension of those results (in [4]) and simplification of their proofs.

### 1. A summary of some existing results.

Recall the following definitions:

DEFINITION 1.1. *The **weak-operator topology**  $w$  on  $L(X, Y)$  is defined by the linear functionals  $T \mapsto f^*(Te)$  for  $f^* \in X^*$  and  $e \in X$ .*

*The **dual weak-operator topology**  $w'$  on  $L(X, Y)$  is defined by the linear functionals  $T \mapsto e^{**}(T^*f^*)$  for  $e^{**} \in X^{**}$  and  $f^* \in Y^*$ . Clearly  $w' \geq w$ .*

THEOREM 1.2. ([10], p.268) *Let  $A$  be a subset of  $K(X, Y)$ . Then  $A$  is weakly compact if and only if  $A$  is  $w'$ -compact.*

THEOREM 1.3. ([10], p.269) *Let  $(T_n)$  be a sequence of compact operators such that  $T_n \rightarrow T$  in  $w'$  where  $T$  is compact. Then  $T_n \rightarrow T$  weakly and there is a sequence  $(S_n)$  of convex combinations of  $\{T_n : n = 1, 2, \dots\}$  with  $\|T - S_n\| \rightarrow 0$ .*

*Remark* ([10], p.269) A Banach space  $X$  is called a Grothendieck space if every weak\*-convergent sequence in  $X^*$  converges weakly in  $X^*$ .  $X$  is a Grothendieck space if and only if for any Banach space  $Y$ , if  $T_n \rightarrow T$  in the weak-operator topology  $w$  on  $K(X, Y)$ , then  $T_n \rightarrow T$  weakly.

We refer to [11], p.29–39, to recall the well known definitions of the *approximation property*, the  *$\lambda$ -bounded approximation property* and the *metric approximation property*.

In his paper [9] J. Johnson proved the following results on projections and imbeddings on dual spaces of  $L(X, Y)$ .

THEOREM 1.4. ([9], p.305) *Let  $X$  and  $Y$  be Banach spaces and suppose  $Y$  has the  $\lambda$ -bounded approximation property. Then there is a projection  $P$  on  $L(X, Y)^*$  such that  $\|P\| \leq \lambda$ , the range of  $P$  is isomorphic to  $K(X, Y)^*$*

(isometric if  $\lambda = 1$ ) and the kernel of  $P$  is the annihilator of  $K(X, Y)$ .

**THEOREM 1.5.** ([9], p.307) *If  $X$  and  $Y$  are Banach spaces and  $Y$  has the  $\lambda$ -bounded approximation property, then there is an isomorphism (isometry if  $\lambda = 1$ ) of  $L(X, Y)$  into  $K(X, Y)^{**}$  whose restriction to  $K(X, Y)$  is the canonical imbedding.*

Several papers have appeared dealing with the question “For which spaces  $X$  and  $Y$  is  $K(X, Y)$  (respectively  $L(X, Y)$ ) reflexive?” The early papers used deep results from Grothendieck’s theory of topological tensor products to attempt answering this question. From the paper of Johnson [9] we have the following result.

**THEOREM 1.6.** ([9], p.307) *Let  $X$  and  $Y$  be Banach spaces, one of which has the approximation property. The following are equivalent:*

- 1)  $K(X, Y)$  is reflexive.
- 2)  $L(X, Y)$  is reflexive.
- 3)  $X, Y$  are reflexive and  $K(X, Y) = L(X, Y)$ .

Refer to [2] for the definitions of the *projective tensor norm* and the *injective tensor norm* on the tensor product of two Banach spaces and recall from [12] (p.72) when a Banach space is said to have *cotype 2*.

**DEFINITION 1.7.** ([8], p.69) *By a Pisier space we will mean an infinite dimensional Banach space  $\mathbf{P}$  such that*

- (1) *on  $\mathbf{P} \otimes \mathbf{P}$  the extremal injective and projective tensor norms are equivalent.*
- (2)  *$\mathbf{P}$  and  $\mathbf{P}^*$  are both cotype 2 spaces.*

Theorem 1.4 cannot be obtained for the space  $X = \mathbf{P}$  and  $Y = \mathbf{P}^*$  using Johnson’s technique, since both  $\mathbf{P}$  and  $\mathbf{P}^*$  lack the approximation property. However, the same result (1.4) was proved by K. John for this case, using the fact that  $T : \mathbf{P} \rightarrow \mathbf{P}^*$  factors through a Hilbert space. Later, in his paper [8], John succeeded in proving a much more general result which includes the case when  $X = \mathbf{P}$  and  $Y = \mathbf{P}^*$ .

K John considered some extensions of Johnson’s result (i.e. of Theorem 1.4) about projections on the dual space of  $L(X, Y)$ , namely:

**THEOREM 1.8.** (cf. John, [7]) *Let  $\mathbf{P}$  be a separable Pisier space. Then the annihilator  $K(\mathbf{P}, \mathbf{P}^*)^\perp$  in the continuous dual space  $L(\mathbf{P}, \mathbf{P}^*)^*$  is the kernel of a projection  $P$  on  $L(\mathbf{P}, \mathbf{P}^*)^*$ . Also the range space of the projection  $P$  is isomorphic to the dual space  $K(\mathbf{P}, \mathbf{P}^*)^*$ .*

THEOREM 1.9. (cf. John, [8]) *Johnson's result (Theorem 1.4) holds for couples of Banach spaces  $X, Y$  such that each bounded linear  $T : X \rightarrow Y$  factors through a Banach space  $Z$ , its dual  $Z^*$  being a separable Banach space which has the bounded approximation property.*

THEOREM 1.10. (cf. John, [8]) *If for each  $T \in L(X, Y)$  there exists a sequence  $(T_n) \subset K(X, Y)$  such that  $T_n \rightarrow T$  in the dual weak operator topology, then the annihilator  $K(X, Y)^\perp$  in  $L(X, Y)^*$  is the kernel of a projection in  $L(X, Y)^*$ .*

The conditions of Theorem 1.10 are for instance satisfied if each  $T \in L(X, Y)$  factors through a Banach space (depending on  $T$ ) which has a separable dual with the approximation property. It should be said that Theorem 1.10 provides information on spaces which are excluded by the conditions (bounded approximation property) in Johnson's result, but it does not provide a strict generalisation of that result. One crucial observation in the proof of Theorem 1.10 is

THEOREM 1.11. (cf. John, [8]) *Suppose the Banach spaces  $X$  and  $Y$  are such that for every  $T \in L(X, Y)$  there is a sequence  $(T_n) \subset K(X, Y)$  such that  $T_n \rightarrow T$  in the dual weak operator topology, then there exists a  $c > 0$  such that for each  $T \in L(X, Y)$  the sequence  $(T_n)$  can be chosen to satisfy  $\|T_n\| \leq c\|T\|$ , for all  $n$ .*

John provides an elegant proof (in [8]) of the fact that with the hypotheses of Theorem 1.10 the norm  $\|T\| := \inf \{\sup_n \|T_n\| : T_n \in K(X, Y), T_n \xrightarrow{w'} T\}$  on  $L(X, Y)$  is equivalent to the uniform norm. This of course implies Theorem 1.11.

## 2. An alternative approach.

In this section we present an alternative approach to John's paper [8]. The main idea in the paper [8] was to present an alternative version to Theorem 1.4 which includes spaces which do not have the approximation property. This study was apparently initiated by John's interest in results on compact operators on Pisier spaces. It seems that examples of (separable) reflexive Pisier spaces are not known yet (or they may not exist at all). This may be the reason why John was not interested in considering Theorem 1.6 in the context of Banach spaces without the approximation property. However, by

now we know that Enflo's example (in 1973) paved the way to showing the existence of many separable reflexive Banach spaces which do not have the approximation property (cf. [6], p.414).

Our aim in the present section is to consider John's results (in more general form) in the environment of operator ideals and then to demonstrate how these results lead to extended versions of results of Kalton and J Johnson about reflexivity of  $K(X, Y)$ . We believe that our presentation does not only provide extensions of existing results but also provides more insight into the structures of the proofs.

**DEFINITION 2.1.** *Let  $T \in L(X, Y)$ .  $T$  is said to have the  $w'$ -compact approximation property ( $w'$ -cap) if there is a sequence  $(T_n) \subset K(X, Y)$  such that  $T_n \xrightarrow{w'} T$ . Let  $\mathcal{L}^{w'}(X, Y)$  be the family of all  $T \in L(X, Y)$  which have the  $w'$ -compact approximation property.*

An easy application of the Uniform Boundedness Theorem shows that

**LEMMA 2.2.** *If  $T_n \rightarrow T$  in the  $w'$ -topology of  $L(X, Y)$  then  $(T_n)$  is norm bounded.*

Let  $X, Y$  be fixed Banach spaces. For  $T \in \mathcal{L}^{w'}(X, Y)$  we put

$$(*) \quad |||T||| = \inf \left\{ \sup_n \|T_n\| : T_n \in K(X, Y), T_n \xrightarrow{w'} T \right\}.$$

Clearly, if  $T \in K(X, Y)$ , then  $|||T||| = \|T\|$ .

Although our following results differ from John's results in [8], we make extensive use of the ideas and techniques developed in that paper.

The reader is referred to [6] and [13] for information in connection with operator ideals. In particular we recall the following criteria for a subclass of the operator ideal  $(L, \|\cdot\|)$  to be a complete operator ideal on the family of all Banach spaces.

**THEOREM 2.3.** (cf. Pietsch [13], 6.2.3, p.91) *Let  $\mathcal{U}$  be a subclass of  $L$  with an  $\mathbb{R}^+$ -valued function  $\alpha$  such that the following conditions are satisfied:*

- (i) *If  $X, Y$  are Banach spaces, then  $a \otimes y \in \mathcal{U}(X, Y)$  for all  $a \in X^*$ ,  $y \in Y$  and  $\alpha(a \otimes y) = \|a\| \|y\|$ .*
- (ii) *If  $S_1, S_2, \dots \in \mathcal{U}(X, Y)$  and  $\sum_{i=1}^{\infty} \alpha(S_i) < \infty$ , then  $S = \sum_{i=1}^{\infty} S_i = \|\cdot\| - \lim_n \sum_{i=1}^n S_i \in \mathcal{U}(X, Y)$ . And  $\alpha(\sum_{i=1}^{\infty} S_i) \leq \sum_{i=1}^{\infty} \alpha(S_i)$ .*
- (iii)  *$RST \in \mathcal{U}(X, Y)$  and  $\alpha(RST) \leq \|R\| \alpha(S) \|T\|$  whenever  $T \in L(X, X_0)$ ,  $S \in \mathcal{U}(X_0, Y_0)$  and  $R \in L(Y_0, Y)$ .*

Then  $(\mathcal{U}, \alpha)$  is a complete normed operator ideal.

This important result is instrumental in proving the following

**THEOREM 2.4.** *Let  $\mathcal{L}^{w'}$  denote the assignment which associates with each pair of Banach spaces  $X, Y$  the vector space  $\mathcal{L}^{w'}(X, Y)$ . And let  $||| \cdot |||$  be the assignment that associates with every pair of Banach spaces  $X, Y$  and with every operator  $S$  belonging to  $\mathcal{L}^{w'}(X, Y)$  the real number  $|||S|||$  in (\*). Then  $(\mathcal{L}^{w'}, ||| \cdot |||)$  is a Banach operator ideal.*

*Proof.* Notice that  $\| \cdot \| \leq ||| \cdot |||$  on  $\mathcal{L}^{w'}(X, Y)$ , where  $\| \cdot \|$  is the uniform operator norm on  $L(X, Y)$ . In fact for any  $\epsilon > 0$ , let  $\|x\| \leq 1$ ,  $\|y^*\| \leq 1$  such that  $\|T\| - \epsilon \leq |y^*(Tx)| = \lim_n |y^*(T_n x)| \leq \sup_n \|T_n\|$  where  $(T_n) \subset K(X, Y)$  such that  $T_n \xrightarrow{w'} T$ . Clearly  $\|T\| \leq |||T||| + \epsilon$ . To prove that  $(\mathcal{L}^{w'}, ||| \cdot |||)$  is a complete normed ideal we make use of Theorem 2.3:

- (i)  $|||I_K||| = 1$  where  $I_K \in \mathcal{L}^{w'}(K)$  is the identity map on the 1-dimensional Banach space  $K$ .
- (iii) Let  $T \in \mathcal{L}(X, X_0)$ ,  $S \in \mathcal{L}^{w'}(X_0, Y_0)$  and  $R \in \mathcal{L}(Y_0, Y)$ . Then if  $S_n \xrightarrow{w'} S$ ,  $S_n \in K(X, Y)$  arbitrary, then  $RS_nT \xrightarrow{w'} RST$ . Hence

$$|||RST||| \leq \sup_n \|RS_nT\| \leq \|R\|(\sup_n \|S_n\|)\|T\|.$$

Since  $(S_n)$  was arbitrarily chosen, it is clear that  $|||RST||| \leq \|R\| |||S||| \|T\|$ .

- (ii) Now suppose that  $(T_n) \subset \mathcal{L}^{w'}(X, Y)$  with  $\sum_{n=1}^{\infty} |||T_n||| < \infty$ . We have to show that  $\sum_{i=1}^{\infty} T_i = \| \cdot \| - \lim_n \sum_{i=1}^n T_i$  exists and is in  $\mathcal{L}^{w'}(X, Y)$  with  $||| \sum_{i=1}^{\infty} T_i ||| \leq \sum_{i=1}^{\infty} |||T_i|||$ : Let  $T_{n,i} \in K(X, Y)$  such that  $T_{n,i} \xrightarrow{w'} T_i$ ,  $\sup_n \|T_{n,i}\| \leq |||T_i||| + \epsilon/2^i$ . For arbitrary  $\|x^{**}\| \leq 1$ ,  $\|y^*\| \leq 1$  we have  $|x^{**}(T_{n,i}^* y^*)| \leq |||T_i||| + \epsilon/2^i$ ,  $\forall i$  and  $\forall n$ . Hence  $\sum_{i=1}^{\infty} x^{**}(T_{n,i}^* y^*)$  converges uniformly in  $n \in \mathbb{N}$ , thus showing that

$$(*) \quad \sum_{i=1}^{\infty} x^{**}(T_i^* y^*) = \lim_n \sum_{i=1}^{\infty} x^{**}(T_{n,i}^* y^*).$$

It follows from the completeness of  $(L(X, Y), \| \cdot \|)$  and  $(K(X, Y), \| \cdot \|)$  and the inequalities  $\|T_i\| \leq |||T_i|||$  for all  $i$  and  $\|T_{n,i}\| \leq |||T_i||| + \epsilon/2^i$  for all  $i$ , that  $\sum_{i=1}^{\infty} T_i \in L(X, Y)$  and  $\sum_{i=1}^{\infty} T_{n,i} \in K(X, Y)$  for all  $n$ . Since (\*) holds for arbitrary  $x^{**} \in B_{X^{**}}$  and  $y^* \in B_{Y^*}$ , it follows that

$\sum_{i=1}^{\infty} T_{n,i} \xrightarrow{w'} \sum_{i=1}^{\infty} T_i$ . Hence  $\sum_{i=1}^{\infty} T_i$  is in  $\mathcal{L}^{w'}(X, Y)$  and

$$\| \sum_i T_i \| \leq \sup_n \left\| \sum_{i=1}^{\infty} T_{n,i} \right\| \leq \sup_n \sum_{i=1}^{\infty} \|T_{n,i}\| \leq \epsilon + \sum_i \|T_i\|.$$

This shows that  $\| \sum_{i=1}^{\infty} T_i \| \leq \sum_{i=1}^{\infty} \|T_i\|$ . By Theorem 2.3,  $(\mathcal{L}^{w'}, \| \cdot \|)$  is a Banach ideal of operators.  $\square$

**THEOREM 2.5.** *There exists a continuous bilinear form  $J : \mathcal{L}^{w'}(X, Y)^* \times \mathcal{L}^{w'}(X, Y) \rightarrow \mathbb{K}$  such that*

- (a)  $J(\phi, T) = \phi(T)$  for all  $(\phi, T) \in \mathcal{L}^{w'}(X, Y)^* \times K(X, Y)$ .
- (b)  $|J(\phi, T)| \leq \|\phi\| \|T\|$  for all  $T \in \mathcal{L}^{w'}(X, Y)$  and  $\phi \in \mathcal{L}^{w'}(X, Y)^*$ .
- (c)  $J(\phi, T) = \lim_n \phi(T_n)$ , where  $(T_n)$  is any sequence of compact operators  $T_n \in K(X, Y)$  tending to  $T$  in  $w'$ -topology.

*Proof.*

- (c) First we observe that if  $T \in \mathcal{L}^{w'}(X, Y)$ ,  $T_n \xrightarrow{w'} T$ ,  $T_n \in K(X, Y)$  for all  $n$ , then  $\lim_n \phi(T_n)$  exists for all  $\phi \in \mathcal{L}^{w'}(X, Y)^*$ : Indeed since  $\{T_n\}_{n \in \mathbb{N}}$  is norm bounded in  $K(X, Y)$ , it is also weakly bounded, i.e.  $\{\phi(T_n)\}_{n \in \mathbb{N}}$  is bounded. Hence, in case of  $\mathbb{K} = \mathbb{R}$ , there are subsequences  $(T_{n_k})$  and  $(T_{m_k})$  such that  $\lim_n \sup \phi(T_n) = \lim_k \phi(T_{n_k})$  and  $\lim_n \inf \phi(T_n) = \lim_k \phi(T_{m_k})$ . Thus  $\lim_n \sup \phi(T_n) - \lim_n \inf \phi(T_n) = \lim_k \phi(T_{n_k} - T_{m_k}) = 0$  because  $T_{n_k} - T_{m_k} \rightarrow 0$  weakly by Theorem 1.3. For the case  $\mathbb{K}$  is complex, we write  $\phi(T_n) = \operatorname{Re}(\phi(T_n)) + i \operatorname{Im}(\phi(T_n))$  and proceed as in the real case. Thus  $\lim_n \phi(T_n)$  exists. Similarly, it follows that if  $T_n \xrightarrow{w'} T$  and  $S_n \xrightarrow{w'} T$  with  $S_n, T_n \in K(X, Y)$  for all  $n$ , then  $\lim_n \phi(T_n) = \lim_n \phi(S_n)$  for all  $\phi \in \mathcal{L}^{w'}(X, Y)^*$ . Thus  $J(\phi, T)$  in (c) is well defined.

- (a)  $J$  is evidently bilinear and if  $T \in K(X, Y)$  then  $J(\phi, T) = \phi(T)$ .
- (b) To prove (b) let  $\phi \in \mathcal{L}^{w'}(X, Y)^*$  and  $T \in \mathcal{L}^{w'}(X, Y)$  be given. Remember that  $\phi$  is continuous on  $K(X, Y)$  with respect to the uniform operator norm. For any  $\epsilon > 0$  there is a sequence  $(T_n) \subset K(X, Y)$  such that  $\sup_n \|T_n\| < (1 + \epsilon) \|T\|$  and  $T_n \xrightarrow{w'} T$ . Hence,

$$\begin{aligned} |J(\phi, T)| &= \left| \lim_n \phi(T_n) \right| \leq \|\phi\| \sup_n \|T_n\| \\ &\leq (1 + \epsilon) \|\phi\| \|T\|. \end{aligned}$$

Since this holds for all  $\epsilon > 0$ , it follows that  $|J(\phi, T)| \leq \|\phi\| \|T\|$ .  $\square$

In the case when  $L(X, Y) = \mathcal{L}^{w'}(X, Y)$ , then since they are both Banach spaces, the norms  $\|\cdot\|$  and  $|||\cdot|||$  are equivalent. Therefore we have

**THEOREM 2.6.** (cf John [8]) *Let  $X, Y$  be Banach spaces such that for every  $T \in L(X, Y)$  there is a sequence  $(T_n) \subset K(X, Y)$  such that  $T_n \xrightarrow{w'} T$ . Then there exists a continuous bilinear form  $J : K(X, Y)^* \times L(X, Y) \rightarrow \mathbb{K}$  and a number  $c > 0$  such that:*

- (a) *If  $T \in K(X, Y)$  and  $\phi \in K(X, Y)^*$ , then  $J(\phi, T) = \phi(T)$ .*
- (b)  *$|J(\phi, T)| \leq c\|\phi\| \|T\|$  for all  $T \in L(X, Y)$  and  $\phi \in K(X, Y)^*$ .*
- (c)  *$J(\phi, T) = \lim_n \phi(T_n)$ , where  $(T_n)$  is any sequence of operators,  $T_n \in K(X, Y)$  tending  $w'$  to  $T$ .*

**COROLLARY 2.7.** *Let  $X, Y$  be Banach spaces. There is a projection  $P : \mathcal{L}^{w'}(X, Y)^* \rightarrow \mathcal{L}^{w'}(X, Y)^*$  such that  $\ker(P) = K(X, Y)^\perp = \{\phi \in \mathcal{L}^{w'}(X, Y)^* : \phi|_{K(X, Y)} = 0\}$  and the range of  $P$  is isomorphic to  $K(X, Y)^*$ .*

*Proof.* Let  $P : \mathcal{L}^{w'}(X, Y)^* \rightarrow \mathcal{L}^{w'}(X, Y)^*$  be defined by  $P\phi(T) = \lim_n \phi(T_n) = J(\phi, T)$  for all  $\phi \in \mathcal{L}^{w'}(X, Y)^*$  and  $T \in \mathcal{L}^{w'}(X, Y)$ . Then  $P$  is well-defined by Theorem 2.5. By the same theorem we have

$$|(P\phi)(T)| = |J(\phi, T)| \leq \|\phi\| \|T\|, \forall T \in \mathcal{L}^{w'}(X, Y).$$

Therefore,  $\|P\phi\| \leq \|\phi\|$ . Hence  $P$  is continuous with  $\|P\| \leq 1$ . It is easily seen that  $P^2 = P$ , i.e. that  $P$  is a projection. Moreover, it is also easy to verify that  $K(X, Y)^\perp \subseteq \ker(P)$ . If we take  $\phi \in \ker(P)$ , then for each  $T \in K(X, Y)$  we have  $0 = P\phi(T) = J(\phi, T) = \phi(T)$  by Theorem 2.5(a). Hence  $\phi(T) = 0, \forall T \in K(X, Y)$ , i.e.  $\phi \in K(X, Y)^\perp$ . We conclude that  $\ker(P) = K(X, Y)^\perp$ . Thus we have Range of  $P \stackrel{\text{isometric}}{=} \mathcal{L}^{w'}(X, Y)^*/K(X, Y)^\perp$ . It is a well known fact (cf [1], p.132, Theorem 2.3) that

$$(K(X, Y), |||\cdot|||)^* \stackrel{\text{isometric}}{=} \mathcal{L}^{w'}(X, Y)^*/K(X, Y)^\perp.$$

Therefore we have

$$\text{Range of } P \stackrel{\text{isometric}}{=} K(X, Y)^*. \quad \square$$

Since the norms  $\|\cdot\|$  and  $|||\cdot|||$  are equivalent when  $L(X, Y) = \mathcal{L}^{w'}(X, Y)$ , it follows from Corollary 2.7 that

**COROLLARY 2.8.** *Let  $X, Y$  be Banach spaces such that for each  $T \in L(X, Y)$  there is a sequence  $(T_n) \subset K(X, Y)$  such that  $T_n \xrightarrow{w'} T$ . Then there*

exists a projection

$$P : L(X, Y)^* \rightarrow L(X, Y)^* \text{ such that}$$

$$\ker(P) = K(X, Y)^\perp = \{\phi \in L(X, Y)^* : \phi|_{K(X, Y)} = 0\}$$

and the range of  $P$  is isomorphic to  $K(X, Y)^*$ .

The bilinear form  $J$  in Theorem 2.5 gives rise to two embedding mappings, which we introduce (without proof) in the following.

**THEOREM 2.9.** *Consider the following two operators:*

- (i)  $J_K : K(X, Y)^* \rightarrow \mathcal{L}^{w'}(X, Y)^*$ , defined by  $(J_K\phi)(T) = J(\tilde{\phi}, T)$  where  $\tilde{\phi} \in \mathcal{L}^{w'}(X, Y)^*$  is any continuous linear extension of  $\phi$ .
- (ii)  $J_{\mathcal{L}} : \mathcal{L}^{w'}(X, Y) \rightarrow K(X, Y)^{**}$ , defined by  $(J_{\mathcal{L}}T)(\phi) = J(\tilde{\phi}, T)$ , where  $\tilde{\phi} \in \mathcal{L}^{w'}(X, Y)^*$  is any continuous linear extension of  $\phi$ .

Then it follows that

- (a)  $J_K$  is an isometry into  $\mathcal{L}^{w'}(X, Y)^*$
- (b)  $J_{\mathcal{L}}$  is a bounded injective linear operator, with  $\|J_{\mathcal{L}}\| \leq 1$ .
- (c)  $J_{\mathcal{L}} = J_K^*|_{\mathcal{L}^{w'}(X, Y)}$
- (d)  $J_{\mathcal{L}}|_{K(X, Y)}$  is the canonical injection of  $K(X, Y)$  into  $K(X, Y)^{**}$ .

*Remark* Let  $T \in \mathcal{L}^{w'}(X, Y)$ . For arbitrary  $\epsilon > 0$  there exist  $x \in X$ ,  $\|x\| \leq 1$  and  $a \in Y^*$ ,  $\|a\| \leq 1$  such that  $\|T\| < |\langle Tx, a \rangle| + \epsilon = \lim_n |\langle T_n x, a \rangle| + \epsilon$  for any sequence  $(T_n) \subset K(X, Y)$  such that  $T_n \xrightarrow{w'} T$ . Let  $\phi(S) = a(Sx)$  for all  $S \in K(X, Y)$ . Then

$$|\phi(S)| = |a(Sx)| \leq \|a\| \|x\| \|S\| \leq \|S\|,$$

showing that  $\phi \in K(X, Y)^*$  and  $\|\phi\| \leq 1$ . Let  $\tilde{\phi}$  be any bounded linear extension of  $\phi$  to  $\mathcal{L}^{w'}(X, Y)^*$  such that  $\|\tilde{\phi}\| \leq 1$ . Then

$$\|T\| < \lim_n |\phi(T_n)| + \epsilon = |\lim_n \phi(T_n)| + \epsilon = |J(\tilde{\phi}, T)| + \epsilon = |J_{\mathcal{L}}(T)(\phi)| + \epsilon.$$

Therefore  $\|T\| \leq \|J_{\mathcal{L}}(T)\|$  for all  $T \in \mathcal{L}^{w'}(X, Y)$ ; thus we have

$$\|T\| \leq \|J_{\mathcal{L}}(T)\| \leq \|T\|$$

for all  $T \in \mathcal{L}^{w'}(X, Y)$ .

Now, if  $X$  and  $Y$  are such that  $\mathcal{L}^{w'}(X, Y) = L(X, Y)$ , then  $\|\cdot\|$  and  $|||\cdot|||$  are equivalent norms. Hence there is a  $c > 0$  such that  $\|T\| \leq \|J_{\mathcal{L}}(T)\| \leq c\|T\|$ . This shows that in this case the linear operator  $J_{\mathcal{L}}$  is also an isomorphism.

**COROLLARY 2.10.** (cf K. John [8], p. 237) *Let  $X, Y$  be Banach spaces such that for each  $T \in L(X, Y)$  there is a sequence  $(T_n) \subset K(X, Y)$  such that  $T_n \xrightarrow{w'} T$ . Then the bilinear form  $J$  in Theorem 2.4 gives rise to the following two isomorphic embeddings:*

$$i) J_K : K(X, Y)^* \rightarrow L(X, Y)^* : (J_K\phi)(T) = J(\tilde{\phi}, T)$$

ii)  $J_L : L(X, Y) \rightarrow K(X, Y)^{**} : (J_L T)(\phi) = J(\tilde{\phi}, T)$  where  $\tilde{\phi} \in L(X, Y)^*$  is any continuous linear extension of  $\phi \in K(X, Y)^*$ .

**LEMMA 2.11.** *Let  $X, Y$  be Banach spaces, with  $Y$  reflexive. Then  $\mathcal{L}^{w'}(X, Y)$  and  $\mathcal{L}^{w'}(Y^*, X^*)$  are isometrically identifiable. In particular,  $T \in \mathcal{L}^{w'}(X, Y)$  if and only if  $T^* \in \mathcal{L}^{w'}(Y^*, X^*)$  and  $|||T||| = |||T^*|||$ .*

*Proof.* Suppose  $T \in \mathcal{L}^{w'}(X, Y)$  and  $(T_n) \subset K(X, Y)$  such that  $T_n \xrightarrow{w'} T$ . Then, for  $y^* \in Y^{***} = Y^*$  and  $x^{**} \in X^{**}$  it is clear that

$$\langle T_n^{**} x^{**}, y^* \rangle = \langle x^{**}, T_n^* y^* \rangle \xrightarrow{n} \langle x^{**}, T^* y^* \rangle = \langle T^{**} x^{**}, y^* \rangle.$$

Thus  $T_n^* \xrightarrow{w'} T^*$  and by Schauder's Theorem  $(T_n^*) \subset K(Y^*, X^*)$ . Hence  $T^* \in \mathcal{L}^{w'}(Y^*, X^*)$  and  $|||T^*||| \leq \sup_n \|T_n^*\| = \sup_n \|T_n\|$ . Since the sequence  $(T_n)$  was arbitrarily chosen, it is clear that

$$(A) \quad |||T^*||| \leq |||T|||.$$

Now let  $S \in \mathcal{L}^{w'}(Y^*, X^*)$  and suppose  $S_n \xrightarrow{w'} S$  where  $(S_n) \subset K(Y^*, X^*)$ . Then each  $T_n = S_n^*|_X$  is compact (from  $X$  into  $Y$ ) and  $T_n^* = S_n$ . Similarly if  $T = S^*|_X$ , then  $T^* = S$ , and

$$\begin{aligned} \langle T_n^* y^*, x^{**} \rangle &= \langle y^*, S_n^* x^{**} \rangle \rightarrow \langle y^*, S^* x^{**} \rangle = \langle T^* y^*, x^{**} \rangle \\ &\text{for all } x^{**} \in X^{**} \text{ and } y^* \in Y^*. \end{aligned}$$

Thus  $T_n \xrightarrow{w'} T$ . Hence  $T \in \mathcal{L}^{w'}(X, Y)$  and  $T^* = S$ . Moreover  $|||T||| \leq \sup_n \|T_n\| = \sup_n \|T_n^*\| = \sup_n \|S_n\|$ . The sequence  $(S_n)$  was arbitrarily chosen. Hence

$$(B) \quad |||T||| \leq |||S||| = |||T^*|||.$$

From (A) and (B) we have  $|||T||| = |||T^*|||$ .

We have thus established that  $T \rightarrow T^*$  defines an isometric isomorphism.  $\square$

We are now ready to consider some results (extending existing ones) on reflexivity of  $K(X, Y)$ .

**THEOREM 2.12.** *Let  $X$  and  $Y$  be Banach spaces. Consider the following statements:*

- (1)  $K(X, Y)$  is reflexive.
- (2)  $\mathcal{L}^{w'}(X, Y)$  is reflexive.
- (3)  $X$  and  $Y$  are reflexive and  $K(X, Y) = \mathcal{L}^{w'}(X, Y)$ .
- (4)  $X$  and  $Y$  are reflexive and  $K(X, Y)$  is  $w$ -closed in  $L(X, Y)$ .

We have (1)  $\iff$  (2)  $\implies$  (3), (4)  $\implies$  (3) and (4)  $\implies$  (1).

*Proof.* (2)  $\implies$  (1) is clear because  $K(X, Y)$  is a closed subspace of  $\mathcal{L}^{w'}(X, Y)$ .

(1)  $\implies$  (2) In this case  $J_{\mathcal{L}} : \mathcal{L}^{w'}(X, Y) \rightarrow K(X, Y)^{**} = K(X, Y)$  (cf. Theorem 2.9) is surjective. By the Open Mapping Theorem  $J_{\mathcal{L}}$  defines an isomorphism. Hence  $\mathcal{L}^{w'}(X, Y)$  is reflexive.

(2)  $\implies$  (3) If  $\mathcal{L}^{w'}(X, Y)$  is reflexive, then  $Y$  is reflexive (because it is isomorphic to a closed subspace of  $K(X, Y)$ ). We have to show that  $X$  is reflexive. Since  $Y$  is reflexive, the spaces  $\mathcal{L}^{w'}(X, Y)$ , and  $\mathcal{L}^{w'}(Y^*, X^*)$  are isometrically identifiable (by Lemma 2.11). Thus  $\mathcal{L}^{w'}(Y^*, X^*)$  is reflexive. As before this implies that  $X^*$  is reflexive. Hence  $X$  is reflexive. Also,  $J_{\mathcal{L}} = J_K^*$  is surjective and an isometry in this case.

(4)  $\implies$  (3) This is clear, since  $w = w'$  in this case.

(4)  $\implies$  (1) Suppose  $X$  and  $Y$  are reflexive and  $\overline{K(X, Y)}^w = K(X, Y)$ . Recall that  $w = w'$  since  $X$  is reflexive. Thus by Kalton (cf. Theorem 1.2) each  $w$ -compact subset of  $K(X, Y)$  is weakly compact. We prove that  $B_{K(X, Y)}$  is  $w$ -compact, hence also weakly compact: Let  $(T_\delta) \subset B_{K(X, Y)}$  be any net. There exists a subnet  $(S_\delta)$  of  $(T_\delta)$  and  $f \in B_{K(X, Y)^{**}}$  such that  $S_\delta \rightarrow f$  in  $wk^*$ -topology. For each  $x \in X$ ,  $y^* \in Y^*$  consider  $y^* \otimes x : K(X, Y) \rightarrow \mathbb{K} : S \rightarrow \langle Sx, y^* \rangle$ , which is a bounded linear functional. We have

$$f(y^* \otimes x) = \lim_{\delta} \langle S_\delta, y^* \otimes x \rangle = \lim_{\delta} \langle S_\delta x, y^* \rangle \quad \forall x \in X, y^* \in Y^*.$$

Let  $T : X \rightarrow Y^{**} = Y$  be defined by  $\langle Tx, y^* \rangle = f(y^* \otimes x)$ ,  $\forall x \in X$

$\forall y^* \in Y^*$ . Then  $T : X \rightarrow Y$  is a bounded linear operator with

$$|\langle Tx, y^* \rangle| = |f(y^* \otimes x)| \leq \|f\| \|y^*\| \|x\|.$$

Clearly,  $\|T\| \leq \|f\| \leq 1$  and  $T = w - \lim_{\delta} S_{\delta}$ . It follows that  $T \in \overline{K(X, Y)}^w = K(X, Y)$  with  $\|T\| \leq 1$ . We have thus shown that any net in  $B_{K(X, Y)}$  has a subnet which converges in  $B_{K(X, Y)}$  with respect to the  $w$ -topology; that is  $B_{K(X, Y)}$  is weakly compact. This proves that  $K(X, Y)$  is reflexive.  $\square$

**COROLLARY 2.13.** *If  $\mathcal{L}^{w'}(X, Y) = \overline{K(X, Y)}^{w'}$  then the following are equivalent:*

- (1)  $K(X, Y)$  is reflexive.
- (2)  $\overline{K(X, Y)}^{w'}$  is reflexive.
- (3)  $X, Y$  are reflexive and  $K(X, Y) = \overline{K(X, Y)}^{w'}$ .

**COROLLARY 2.14.** *If  $\mathcal{L}^{w'}(X, Y) = L(X, Y)$ . Then the following are equivalent:*

- (1)  $K(X, Y)$  is reflexive.
- (2)  $L(X, Y)$  is reflexive.
- (3)  $X, Y$  are reflexive and  $K(X, Y) = L(X, Y)$ .

*Remark:* Although (3)  $\implies$  (2) in Theorem 1.6 was proved by Kalton (cf. Kalton [10]) and independently by Kheinrich without assuming the presence of the approximation property, it was mentioned in Johnson's paper [9] that it is apparently still open whether (2)  $\implies$  (3), in the same theorem, is true without assuming the presence of the approximation property. Our result now shows that the implication is still true if we have the assumption  $L(X, Y) = \mathcal{L}^{w'}(X, Y)$ , where it may be that neither  $X$  nor  $Y$  has the approximation property.

**LEMMA 2.15.** *Suppose  $X$  and  $Y$  are Banach spaces such that  $X^{**}$  and  $Y^*$  are separable. Then the closed unit ball  $B_{L(X, Y)}$  is  $w'$ -metrisable.*

*Proof.* Let  $A := \{e_n^{**} : n \in \mathbb{N}\}$  and  $B := \{f_m^* : m \in \mathbb{N}\}$  be countable dense subsets of  $X^{**}$  and  $Y^*$  respectively. We show that the countable family of continuous seminorms

$$T \mapsto |e_n^{**}(T^* f_m^*)|, \forall m, n \in \mathbb{N}$$

generates the  $w'$  topology on  $B_{L(X, Y)}$ : Let  $(T_{\delta})$  be a net in  $B_{L(X, Y)}$  such that  $|e_n^{**}(T_{\delta}^* f_m^*)| \xrightarrow{\delta} 0$  for each pair  $(e_n^{**}, f_m^*)$ . Consider arbitrary  $e^{**} \in X^{**}$  and

$f^* \in Y^*$ . For  $\epsilon > 0$  given, let  $e_{n_0}^{**} \in A$  such that  $\|e^{**} - e_{n_0}^{**}\| < \epsilon/(3\|f^*\|)$ . Then let  $f_{m_0}^* \in B$  such that  $\|f^* - f_{m_0}^*\| < \epsilon/(3\|e_{n_0}^{**}\|)$ . Let  $\delta_0$  be an index such that  $|e_{n_0}^{**}(T_\delta^* f_{m_0}^*)| < \epsilon/3$  for all  $\delta \geq \delta_0$ . We have

$$\begin{aligned} |e^{**}(T_\delta^* f^*)| &\leq |e^{**}(T_\delta^* f^*) - e_{n_0}^{**}(T_\delta^* f^*)| \\ &\quad + |e_{n_0}^{**}(T_\delta^* f^*) - e_{n_0}^{**}(T_\delta^* f_{m_0}^*)| + |e_{n_0}^{**}(T_\delta^* f_{m_0}^*)| \\ &\leq \|e^{**} - e_{n_0}^{**}\|(\sup_\delta \|T_\delta^* f^*\|) \\ &\quad + \|e_{n_0}^{**}\| \|T_\delta^* f^* - T_\delta^* f_{m_0}^*\| + |e_{n_0}^{**}(T_\delta^* f_{m_0}^*)| \\ &< \epsilon, \quad \forall \delta \geq \delta_0. \end{aligned}$$

Having a countable neighbourhood basis of the origin, it follows that the Hausdorff topology  $w'$  on  $B_{L(X,Y)}$  is metrisable (cf. [15], Theorem 4, p.16).  $\square$

In the setting of separable Banach spaces we have the following version of Theorem 1.6:

**COROLLARY 2.16.** *Let  $X$  and  $Y$  be separable Banach spaces. The following are equivalent:*

- (1)  $K(X, Y)$  is reflexive.
- (2)  $\mathcal{L}^{w'}(X, Y)$  is reflexive.
- (3)  $X, Y$  are reflexive and  $K(X, Y) = \mathcal{L}^{w'}(X, Y)$ .

*Proof.* We only have to check that (3)  $\implies$  (1) holds. Refer to the proof of (4)  $\implies$  (1) in (2.12). Following the same arguments it follows for a given net  $(T_\delta) \subset B_{K(X,Y)}$  that a subnet  $(S_\gamma) \subset B_{K(X,Y)}$  converges to a  $T \in \overline{K(X, Y)}^{w'}$  with  $\|T\| \leq 1$ . Thus  $T$  is in the closure of  $B_{K(X,Y)}$  in the metrisable space  $(B_{L(X,Y)}, w')$  (refer to Lemma 2.15). Hence there is a sequence  $(T_n) \subset B_{K(X,Y)}$  such that  $T_n \xrightarrow{w'} T$ . Thus  $T \in \mathcal{L}^{w'}(X, Y) = K(X, Y)$ . Therefore,  $B_{K(X,Y)}$  is  $w'$ -compact and hence weakly compact.

### 3. Extensions to locally convex spaces

Let  $E$  and  $F$  denote Hausdorff locally convex spaces. The algebraic dual space of a linear space  $E$  is denoted by  $E'$ .  $E_\beta^*$  will denote the continuous dual space  $E^*$  of  $E$  when it carries the strong topology  $\beta(E^*, E)$ . However, if not otherwise specified, the topology on  $E^*$  will throughout the section be assumed to be the strong topology. The bidual is defined by  $E^{**} = (E_\beta^*)^*$ . The space  $L(E, F)$  of continuous linear operators is a Hausdorff locally convex

space when it is endowed with the topology of uniform convergence on the bounded sets in  $E$ , in which case it is denoted by  $L_b(E, F)$ . The topology is generated by zero neighbourhoods of the form

$$[A, V] := \{T \in L(E, F) : T(A) \subseteq V\}$$

where  $V$  and  $A$  run through a zero neighbourhood basis  $\mathcal{U}(F)$  of closed absolutely convex sets in  $F$  and a fundamental system  $\mathcal{B}(E)$  of absolutely convex closed bounded sets in  $E$  respectively. The gauge function of  $[A, V]$  is denoted by  $P_{A,V}$ .

DEFINITION 3.1. *An operator  $T \in L(E, F)$  is said to be quasi compact if  $T(A)$  is precompact in  $F$  for each bounded set  $A$  in  $E$ . The vector space of quasi compact operators in  $L(E, F)$  is denoted by  $K(E, F)$ . We use the notation  $K_b(E, F)$  when it carries the subspace topology of uniform convergence on the bounded sets in  $E$ .*

The existence of a (continuous with respect to the strong topology) projection on  $L_b(E, F)^*$  with kernel the annihilator  $K(E, F)^\perp$  and its applications to topological decomposition of the bidual space  $K_b(E, F)^{**}$  and results on semi-reflexivity of  $L_b(E, F)$  are considered in the papers [3] and [5]. The results in the paper [5] are generalisations of those in [3] and [9]. Especially in [5] the following generalisations of Johnson's result are proved.

THEOREM 3.2. (cf. [5], Theorem 2.2) *There exists a (continuous with respect to the strong topology) projection  $P : L_b(E, F)^* \rightarrow L_b(E, F)^*$  with kernel  $K(E, F)^\perp$  if either*

- (a)  *$F$  has the quasi compact approximation property, that is there exists an equicontinuous net  $(T_\delta) \subset K_b(F, F)$  such that  $T_\delta x \rightarrow x$  in  $F$  for each  $x \in F$ , or*
- (b)  *$E$  has the shrinking quasi compact approximation property, that is there exists an equicontinuous net  $(T_\delta) \subset K_b(E, E)$  such that  $T_\delta x \rightarrow x$  for each  $x \in E$  and  $T_\delta^* x^* \rightarrow x^*$  strongly for each  $x^* \in E^*$ .*

Motivated by John's paper [8], Fourie (in [4]) studied projections on dual spaces of spaces of operators in the locally convex space setting. Before we take a closer look at the results obtained in [4], we consider a locally convex version of Kalton's theorem (Theorem 1.3).

Define  $w'$  on  $L(E, F)$  as in the Banach space case. Hence, a net  $(T_\delta)$  converges  $w'$  to  $T$  in  $L(E, F)$  if

$$x^{**}(T_\delta^* y^*) \rightarrow x^{**}(T^* y^*)$$

for all  $x^{**} \in E^{**}$  and  $y^* \in F^*$ .

For the proof of the following generalisation of Kalton's result to the locally convex setting, we refer to [4]:

**THEOREM 3.3.** ([4], Proposition 1) *Suppose that  $(T_n)$  is a bounded sequence in  $K_b(E, F)$  which converges in the dual weak operator topology to  $T \in K(E, F)$ . Then  $\phi(T_n) \rightarrow \phi(T)$  for all  $\phi \in L_b(E, F)^*$ .*

This generalisation of Kalton's result will allow us to follow our alternative approach in order to extend our results in the previous section to the locally convex space setting. Once we have the above "generalised version" of Kalton's result, it is a matter of mimicking John's argument in [8] to show that if a bounded sequence  $(T_n) \subset K(E, F)$  converges to  $T \in L(E, F)$  with respect to the dual weak operator topology, then  $\lim_n \phi(T_n)$  exists for each  $\phi \in L_b(E, F)^*$ . The idea is to take subsequences  $(\phi(T_{n_k}))$  and  $(\phi(T_{m_k}))$  such that

$$\begin{aligned}\alpha_0 &= \limsup_n \phi(T_n) = \lim_k \phi(T_{n_k}) \\ \beta_0 &= \liminf_n \phi(T_n) = \lim_k \phi(T_{m_k})\end{aligned}$$

and then to notice that since the sequence  $(T_{n_k} - T_{m_k})$  is bounded and converges to 0 in the dual weak operator topology, we have that

$$\alpha_0 - \beta_0 = \lim_k \phi(T_{n_k} - T_{m_k}) = 0$$

by (3.3). A similar argument shows that the limit  $\lim_n \phi(T_n)$  is independent of the choice of the bounded sequence  $(T_n)$ . Thus if each  $T \in L(E, F)$  is the  $w'$ -limit of a bounded sequence  $(T_n)$  in  $K_b(E, F)$ , then the operator

$$\begin{aligned}P : L_b(E, F)^* &\rightarrow L_b(E, F)' \\ \phi &\rightarrow P\phi : P\phi(T) = \lim_n \phi(T_n)\end{aligned}$$

into the algebraic dual space  $L_b(E, F)'$  is well defined.

It has to be decided whether  $P$  maps into the continuous dual space  $L_b(E, F)^*$ . Using the Riesz representation theorem it is shown in [4] that if each  $T \in L_b(E, F)$  is the  $w'$ -limit of a bounded sequence  $(T_n) \subset K_b(E, F)$ , then this is true. Hence we have

**THEOREM 3.4.** ([4], Proposition 2) *Suppose that each  $T \in L_b(E, F)$  is*

the  $w'$ -limit of a bounded sequence  $(T_n) \subset K_b(E, F)$ . Then the linear operator

$$P : L_b(E, F)^* \rightarrow L_b(E, F)' : P\phi(T) = \lim_n \phi(T_n)$$

maps into  $L_b(E, F)^*$  and is a closed graph projection with kernel  $K_b(E, F)^\perp$ . In particular,  $K_b(E, F)^*$  and  $K_b(E, F)^\perp$  are isomorphic to closed (in the strong topology) subspaces of  $L_b(E, F)^*$ .

**COROLLARY 3.5.** *Let  $X$  and  $Y$  be normed spaces. If for each  $T \in L(X, Y)$  there exists a bounded sequence  $(T_n) \subset K(X, Y)$  such that  $T_n \rightarrow T$  in the dual weak operator topology, then the annihilator  $K(X, Y)^\perp$  in  $L(X, Y)^*$  is the kernel of a projection on  $L(X, Y)^*$ . In particular if  $X$  is  $\aleph_0$ -barrelled, then the hypothesis that the sequence  $(T_n)$  should be norm bounded can be ignored (cf [6], p.252).*

Moreover, the different versions of the closed graph theorem for locally convex spaces and Theorem 3.3 provide more general versions of Theorem 3.4 such that the projection mapping  $P$  is indeed continuous. In particular we have

**COROLLARY 3.6.** *If for instance  $L(E, F)$  is a  $df$ -space, then  $L(E, F)^*$  with the strong topology is a Fréchet space (cf. [6], p 257) so that with the hypotheses of Theorem 3.4, the mapping  $P$  is continuous.*

We intend to show that by following our “alternative approach”, as was outlined in the previous section in the Banach space setting, one can easily obtain and extend the above result (Theorem 3.4). In the proof of Theorem 3.4 (as is discussed in [4]) one has to make use of the Riesz representation theorem and the Lebesgue Dominated Convergence Theorem in order to show that the projection  $P$  maps into  $L_b(E, F)^*$ . Following is an alternative approach which results in both simplification and extension of Theorem 3.4 and its proof.

Denote by  $\mathcal{L}^{w'}(E, F)$  the vector space of all  $T \in L(E, F)$  such that  $T_n \xrightarrow{w'} T$  for some bounded sequence  $(T_n) \subset K_b(E, F)$ . On  $\mathcal{L}^{w'}(E, F)$  we define a locally convex topology by means of the seminorms

$$\pi_{A,V}(T) = \inf_n \{ \sup_n P_{A,V}(T_n) : T_n \in K(E, F), T_n \xrightarrow{w'} T \},$$

where  $A \in \mathcal{B}(E)$  and  $V \in \mathcal{U}(F)$ . Put

$$W_{A,V} = \{ T \in \mathcal{L}^{w'}(E, F) : \pi_{A,V}(T) \leq 1 \}$$

and let

$$q_{A,V}(\phi) = \sup\{|\phi(S)| : S \in W_{A,V}\}$$

for every (continuous) linear functional  $\phi$  on  $\mathcal{L}^{w'}(E, F)$ . Clearly  $\phi$  is bounded on  $W_{A,V}$  if and only if  $q_{A,V}(\phi) < \infty$ .

**THEOREM 3.7.** *Fix the locally convex spaces  $E$  and  $F$ . Let  $\mathcal{L}^{w'} = \mathcal{L}^{w'}(E, F)$ ,  $K = K(E, F)$  and  $\mathcal{L}^* = \mathcal{L}^{w'}(E, F)^*$ . There exists a bilinear form  $J : \mathcal{L}^* \times \mathcal{L}^{w'} \rightarrow \mathbb{K}$  such that*

- (a)  $J(\phi, T) = \phi(T)$  for all  $(\phi, T) \in \mathcal{L}^* \times K$ .  
 (b) For all  $T \in \mathcal{L}^{w'}$ ,  $\phi \in \mathcal{L}^*$  and for all  $A \in \mathcal{B}(E)$ ,  $V \in \mathcal{U}(F)$ , we have

$$|J(\phi, T)| \leq q_{A,V}(\phi)\pi_{A,V}(T).$$

- (c)  $J(\phi, T) = \lim_n \phi(T_n)$ , where  $(T_n)$  is any bounded sequence of operators  $T_n \in K_b(E, F)$  tending to  $T$  in  $w'$ -topology.

*Proof.* (c) The proof is similar to the proof of (c) in Theorem 2.5, this time using that each  $T \in \mathcal{L}^{w'}$  is the  $w'$ -limit of a bounded sequence  $(T_n)$  in  $K_b(E, F)$  and the fact that the same sequence converges weakly to  $T$  (as is proved in Theorem 3.3).

(b) Let  $\phi \in \mathcal{L}^*$  and  $T \in \mathcal{L}^{w'}(E, F)$  be given. Suppose  $\phi$  is bounded on  $W_{A,V}$ . For any  $\epsilon > 0$  there is a bounded sequence  $(T_n) \subset K_b(E, F)$  such that  $\sup_n P_{A,V}(T_n) < (1 + \epsilon)\pi_{A,V}(T)$  and  $T_n \xrightarrow{w'} T$ . Hence,

$$\begin{aligned} |J(\phi, T)| &= \left| \lim_n \phi(T_n) \right| \leq q_{A,V}(\phi) \sup_n P_{A,V}(T_n) \\ &\leq (1 + \epsilon)q_{A,V}(\phi)\pi_{A,V}(T). \end{aligned}$$

Since this holds for all  $\epsilon > 0$  it follows that  $|J(\phi, T)| \leq q_{A,V}(\phi)\pi_{A,V}(T)$ .  $\square$

We are now ready to discuss an alternative result for Theorem 3.4, which has easier proof and which is formulated in more general context.

**PROPOSITION 3.8.** *Let  $E$  and  $F$  be locally convex spaces. The linear operator*

$$P : \mathcal{L}^{w'}(E, F)^* \rightarrow \mathcal{L}^{w'}(E, F)' : P\phi(T) = \lim_n \phi(T_n),$$

where  $T \in \mathcal{L}^{w'}(E, F)$  is the  $w'$ -limit of the bounded sequence  $(T_n) \subset K_b(E, F)$ , maps into  $\mathcal{L}^{w'}(E, F)^*$  and is a closed graph projection with kernel

$K_b(E, F)^\perp$ . In particular,  $K_b(E, F)^*$  and  $K_b(E, F)^\perp$  are isomorphic to closed (in the strong topology) subspaces of  $\mathcal{L}^{w'}(E, F)^*$ .

*Proof.* Let  $\phi \in \mathcal{L}^{w'}(E, F)^*$ . Suppose  $\phi$  is bounded on  $W_{A,V}$  with  $|\phi(S)| \leq 1$  for all  $S \in W_{A,V}$ . From the proof of Proposition 3.7 (b) it follows that  $|J(\phi, T)| \leq \pi_{A,V}(T)$  for all  $T \in \mathcal{L}^{w'}(E, F)$ . This shows (cf. also (3.7)(c)) that  $P\phi$  is continuous, hence that  $P$  maps into  $\mathcal{L}^{w'}(E, F)^*$ . Clearly,  $P^2 = P$ .

The argument to show that  $P$  has closed graph is similar to that used in the proof of Theorem 3.4. For the sake of completeness, we discuss the proof of this fact: Let  $G(P)$  denote the graph of  $P$  and let  $(\phi, \theta) \in \overline{G(P)} \subseteq \mathcal{L}^{w'}(E, F)^* \times \mathcal{L}^{w'}(E, F)^*$  with the product topology defined by the strong topology. Hence there is a net  $\phi_\delta \subset \mathcal{L}^{w'}(E, F)^*$  such that  $\phi_\delta \rightarrow \phi$  and  $P\phi_\delta \rightarrow \theta$  in the strong topology of  $\mathcal{L}^{w'}(E, F)^*$ . Take any  $T \in \mathcal{L}^{w'}(E, F)$ . Let  $\epsilon > 0$  be given. We show that  $|P\phi(T) - \theta(T)| < \epsilon$ :

Let  $\delta_0$  be an index such that  $|P\phi_\delta(T) - \theta(T)| < \epsilon/2$ ,  $\forall \delta \geq \delta_0$ . The set  $\{T_n | n \in \mathbb{N}\}$  is bounded in  $K_b(E, F)$ , therefore it is clear from  $[A, V] \cap K_b(E, F) \subseteq W_{A,V} \cap K_b(E, F)$  for all  $A \in \mathcal{B}(E)$  and  $V \in \mathcal{U}(F)$  that it is also bounded in  $\mathcal{L}^{w'}(E, F)$ ; thus  $\phi_\delta \rightarrow \phi$  uniformly on  $\{T_n\}$ . Let  $\delta_1 \geq \delta_0$  such that  $|\phi(T_n) - \phi_{\delta_1}(T_n)| < \epsilon/2$ ,  $\forall \delta \geq \delta_1$  and  $\forall n$ . Hence

$$|\phi(T_n) - \theta(T)| \leq |\phi(T_n) - \phi_{\delta_1}(T_n)| + |\phi_{\delta_1}(T_n) - \theta(T)| < \epsilon/2 + |\phi_{\delta_1}(T_n) - \theta(T)|,$$

and

$$\begin{aligned} |P\phi(T) - \theta(T)| &= \lim_n |\phi(T_n) - \theta(T)| \\ &\leq \epsilon/2 + \lim_n |\phi_{\delta_1}(T_n) - \theta(T)| \leq \epsilon/2 + |P\phi_{\delta_1}(T) - \theta(T)| \\ &< \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

It follows that  $P\phi(T) = \theta(T)$ ,  $\forall T \in \mathcal{L}^{w'}(E, F)$ , showing that  $P\phi = \theta$ .  $\square$

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