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Assessment of Mathematical Model for Predicting Climate Change Impact on Water Availability and Quality: A SWAT-SEIR Framework

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Abstract-* *This paper critically reviews and assesses various mathematical and computational modeling techniques applied to analyze the impacts of climate change on water security. Emphasis is placed on integrating dynamic hydrological processes with contamination propagation under changing climatic conditions. The study employs a SWAT-SEIR-type model, analyzed by both analytical and numerical methods like the Fourth-Order Runge-Kutta (RK4) scheme, simulate water system dynamics and evaluate stability conditions, particularly focusing on the basic reproduction number $R_{0\ as}$ an index of contamination spread. The analysis contributes to understanding the complex interplay between climate-induced stressors and water resources, providing a foundation for developing robust water security strategies in the face of climate change.

Indexed Terms-* *Climate Change, Water Security, Contamination Propagation, Hydrological Modeling, Basic Reproduction Number, RK4

I. INTRODUCTION

Water security is increasingly threatened by climate change through alterations in precipitation patterns, temperature regimes, and extreme weather events. Understanding the dynamic responses of water systems to these climate-induced changes is vital for sustainable management and mitigation strategies. Modeling techniques serve as crucial tools for simulating water resource dynamics and contamination processes, enabling policymakers and stakeholders to anticipate risks and design adaptive interventions. This study evaluates common mathematical models used to analyze the impact of climate variability and change on water security. It explores the adaptation of epidemiological SEIR frameworks to hydrological contamination dynamics, where compartments represent clean and contaminated water states. The use of numerical methods, particularly the Fourth-Order Runge-Kutta method for solving ordinary differential equations, is detailed to ensure accurate and stable simulations over time. Further, the concept of a Contamination Propagation Index, derived from the basic reproduction number R_0, is introduced as a metric for assessing contamination persistence or decay within water systems. Stability analysis of the contamination-free equilibrium provides insights into critical thresholds for intervention. The integration of these modeling approaches offers a comprehensive assessment of water security challenges driven by climate change.

We consider the SWAT-SEIR compartmental model describing water contamination dynamics influenced by climatic processes:

$$\begin{cases} \dot{S} = P - \alpha S I - \mu_S S, \\ \dot{E} = \alpha S I - \beta E - \mu_E E, \\ \dot{I} = \beta E + PL - \gamma I - \delta I - \mu_I I, \\ \dot{R} = \gamma I + \delta I - \mu_R R, \end{cases} \tag{1}$$

where all parameters and variables $S, E, I, R, P, \alpha, \beta, P\,L, \gamma, \delta, \mu S\ , \mu E\ , \mu I\ , \mu R$ are positive constants or time-dependent functions as described in the model formulation.

II. BASIC REPRODUCTION NUMBER R_0

Based on the SWAT-SEIR model for climate-driven water security dynamics, we employ the next generation matrix method to compute the basic reproduction number R_0, herein interpreted as the Contamination Propagation Index. R_0 measures the average number of secondary contamination events generated by one unit of contaminated water introduced into a clean aquatic system.

The infected compartments are:

$$\mathbf{x} = \begin{bmatrix} E \\ I \end{bmatrix}$$

(2)

We rewrite the system in the standard form:

$$\frac{d\mathbf{x}}{dt} = \mathcal{F}(\mathbf{x}) - \mathcal{V}(\mathbf{x}),$$

(3)

where:

$\mathcal{F}(\mathbf{x})$ denotes new contamination inputs,
$\mathcal{V}(\mathbf{x})$ includes all transitions and removals.

From the model equation (1)

$$\mathcal{F}(\mathbf{x}) = \begin{bmatrix} \alpha S I \\ 0 \end{bmatrix}, \qquad \mathcal{V}(\mathbf{x}) = \begin{bmatrix} \beta E + \mu_E E \\ -\beta E + (\gamma + \delta + \mu_I) I - PL \end{bmatrix}$$

At the Contamination-free equilibrium (DFE),

$$\mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad S = S^* = \frac{P}{\mu_S},$$

assuming $\alpha I \approx 0$ at DFE.

From Linearization and Jacobians at DFE

$$F = \left.\frac{\partial \mathcal{F}}{\partial \mathbf{x}}\right|_{\text{DFE}} = \begin{bmatrix} \frac{\partial}{\partial E}(\alpha S I) & \frac{\partial}{\partial I}(\alpha S I) \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \alpha S^* \\ 0 & 0 \end{bmatrix}$$

(3)

Similarly

$$V = \left.\frac{\partial \mathcal{V}}{\partial \mathbf{x}}\right|_{\text{DFE}} = \begin{bmatrix} \beta + \mu_E & 0 \\ -\beta & \gamma + \delta + \mu_I \end{bmatrix}$$

(4)

Next-Generation Matrix $K = F\,V^{-1}$

Let

$$V = \begin{bmatrix} a & 0 \\ -b & d \end{bmatrix}, \quad a = \beta + \mu_E, \quad b = \beta, \quad d = \gamma + \delta + \mu_I.$$

The inverse of V is

$$V^{-1} = \frac{1}{ad} \begin{bmatrix} d & 0 \\ b & a \end{bmatrix}$$

(5)

Therefore

$$K = FV^{-1} = \frac{1}{ad} \begin{bmatrix} 0 & \alpha S^* \\ 0 & 0 \end{bmatrix} \begin{bmatrix} d & 0 \\ b & a \end{bmatrix} = \frac{1}{ad} \begin{bmatrix} \alpha S^* b & \alpha S^* a \\ 0 & 0 \end{bmatrix}$$

Explicitly

$$K = \begin{bmatrix} \frac{\alpha \beta S^*}{(\beta + \mu_E)(\gamma + \delta + \mu_I)} & \frac{\alpha S^*}{\gamma + \delta + \mu_I} \\ 0 & 0 \end{bmatrix}$$

(6)

From eigenvalues of K

$$\lambda_1 = 0, \quad \lambda_2 = \frac{\alpha \beta S^*}{(\beta + \mu_E)(\gamma + \delta + \mu_I)}.$$

(7)

The spectral radius (dominant eigenvalue) is thus

$$\boxed{R_0 = \frac{\alpha \beta S^*}{(\beta + \mu_E)(\gamma + \delta + \mu_I)}.}$$

(8)

which is the Basic Reproduction Number R_0

A. Physical Interpretation of R_0

If $R_0<1$, contamination cannot sustain itself and dies out asymptotically.

If $R_0 > 1$, contamination persists and possibly grows, indicating the necessity for intervention.

R_0 depends on: the exposure rate α, the progression rate β, the equilibrium clean water volume S^*, the removal and recovery rates γ, δ, μ_E, μ_I.

B. Contamination Free Equilibrium (CFE)

The DFE in the context of hydrological models represents a state where the water system is free from contamination. It occurs when all contaminant concentrations are zero and the system is in a steady state. The CFE (Contamination-Free Equilibrium) in the SWAT-SEIR model would occur when the rates of change for all compartments are zero. The specific values of the compartments at the CFE depend on the parameter values and initial conditions of the model.

At the Contamination-Free Equilibrium (CFE):

$$E^* = 0, \; I^* = 0, \; R^* = 0.$$

Setting derivatives at steady state $dS/dt = 0$ with $I^* = 0$ gives:

$$P - \alpha S^* \cdot 0 - \mu_S S^* = 0 \implies S^* = \frac{P}{\mu_S}.$$

Thus the CFE is:

$$(S^*, E^*, I^*, R^*) = \left(\frac{P}{\mu_S}, 0, 0, 0\right).$$

(9)

This equilibrium represents a clean and uncontaminated water system. It is a critical reference point used to evaluate the potential for contamination to emerge in the system, which will be explored using the basic reproduction number R_0 and local stability analysis of the Jacobian matrix.

C. Local Stability of the Contamination-Free Equilibrium

To analyze the local stability of the CFE, we linearize the system at the equilibrium to give the Jacobian matrix J, with entries defined as:

$$J = \left[\frac{\partial f_i}{\partial x_j}\right]_{(S,E,I,R)=(S^*,0,0,0)} = \begin{bmatrix} -\mu_S & 0 & -\alpha S^* & 0 \\ 0 & -(\beta + \mu_E) & \alpha S^* & 0 \\ 0 & \beta & -(\gamma + \delta + \mu_I) & 0 \\ 0 & 0 & \gamma + \delta & -\mu_R \end{bmatrix}$$

10)

Note:
The off-diagonal elements αS^* and β induce coupling between compartments, making the matrix not to be strictly lower triangular.

Because J exhibits a block lower-triangular form, the eigenvalues consist of:

One eigenvalue from the top-left element:

$\lambda_1 = -\mu S < 0.$

One eigenvalue from the bottom-right element:

$\lambda_4 = -\mu R < 0.$

Two eigenvalues λ_2, λ_3 are roots of the 2 × 2 submatrix:

$$J_{\text{sub}} = \begin{bmatrix} -(\beta+\mu_E) & \alpha S^* \\ \beta & -(\gamma+\delta+\mu_I) \end{bmatrix} \tag{11}$$

The characteristic polynomial satisfies:

$det(J_{sub} - \lambda_I) = 0,$

leading to:

$(\lambda + \beta + \mu_E)(\lambda + \gamma + \delta + \mu_I) - \alpha\beta S^* = 0,$

or equivalently,

$\lambda^2 + \lambda(\beta + \mu_E + \gamma + \delta + \mu_I) + (\beta + \mu_E)(\gamma + \delta + \mu_I) - \alpha\beta S^* = 0$

For local stability, all eigenvalues must have negative real parts. Since all parameters are positive, the sign of the real parts depends on the constant term:

$(\beta + \mu_E)(\gamma + \delta + \mu_I) - \alpha\beta S^* > 0,$

which rearranges to

$$R_0 = \frac{\alpha\beta S^*}{(\beta+\mu_E)(\gamma+\delta+\mu_I)} < 1 \tag{12}$$

Hence, the CFE is locally asymptotically stable if and only if $R_0 < 1$.

D. Trace and Determinant of J

Trace is given by:

$Tr(J) = -\mu_S - (\beta + \mu_E) - (\gamma + \delta + \mu_I) - \mu_R < 0,$

since all parameters are positive.

For determinant, we compute via cofactor expansion along the first row.

The first row is:

$a_{11} = -\mu_S$, $a_{12} = 0$, $a_{13} = -\alpha S^*$, $a_{14} = 0$.

The determinant is given by:

$$\det(J) = \sum_{j=1}^{4} (-1)^{1+j} a_{1j} \det(M_{1j}),$$

where M_{1j} is the minor formed by deleting the first row and *j-th* column.

Due to zero elements except at positions *1* and *3*,

$det(J) = a_{11}C_{11} + a_{13}C_{13},$

where

$C_{1j} = (-1)1+j\ det(M_{1j}).$

Calculating the cofactors:

-For C_{11}:

The minor matrix M_{11} is

$$M_{11} = \begin{bmatrix} -(\beta+\mu_E) & \alpha S^* & 0 \\ \beta & -(\gamma+\delta+\mu_I) & 0 \\ 0 & \gamma+\delta & -\mu_R \end{bmatrix}$$

The sign factor is $(-1)1+1 = +1$, so

$C_{11} = +\ det(M_{11}).$

- For C13:

The minor matrix M_{13} is

$$M_{13} = \begin{bmatrix} 0 & -(\beta+\mu_E) & 0 \\ 0 & \beta & 0 \\ 0 & 0 & -\mu_R \end{bmatrix}$$

The sign factor is $(-1)1+3 = +1$, so

$C_{13} = +\ det(M_{13}).$

Observe that the first column of M_{13} consists entirely of zeros:

So $det(M_{13}) = 0$.

Therefore,

$det(J) = a_{11}C_{11} = (-\mu_S) \times det(M_{11}).$

Computing $det(M_{11})$

Note that the third column of M_{11} contains only one nonzero element $-\mu_R$ at row 3, column 3.

Expand $det(M_{11})$ along the third column:

$$\det(M_{11}) = (-1)^{3+3}(-\mu_R) \times \det \begin{bmatrix} -(\beta+\mu_E) & \alpha S^* \\ \beta & -(\gamma+\delta+\mu_I) \end{bmatrix} = -\mu_R \times D,$$

where the 2 × 2 determinant D is

$$D = \begin{vmatrix} -(\beta+\mu_E) & \alpha S^* \\ \beta & -(\gamma+\delta+\mu_I) \end{vmatrix} = (\beta+\mu_E)(\gamma+\delta+\mu_I) - \alpha\beta S^*.$$

Final expression for the determinant by substituting back,

$$\det(J) = (-\mu_S) \times (-\mu_R) \times D = \mu_S\mu_R[(\beta+\mu_E)(\gamma+\delta+\mu_I) - \alpha\beta S^*].$$

All parameters μ_S, μ_R, β, μ_E, γ, δ, μ_I are positive,

so $\mu_S\,\mu_R > 0$.

The sign of $det(J)$ depends on

$N := (\beta+\mu_E)(\gamma+\delta+\mu_I) - \alpha\beta S^*.$

The classical basic reproduction number is defined as

$$\boxed{R_0 := \frac{\alpha\beta S^*}{(\beta+\mu_E)(\gamma+\delta+\mu_I)}}.$$

The determinant is positive if and only if

$N > 0 \Leftrightarrow R_0 < 1$

E.Global Stability of the Contamination-Free Equilibrium (CFE)

Consider the SWAT-SEIR water contamination model with compartments:

$x = \begin{bmatrix} S \\ R \end{bmatrix}$ (Uninfected compartments)

$y = \begin{bmatrix} E \\ I \end{bmatrix}$ (Infected compartments)

The system can be rewritten in the form

$$\begin{cases} \dfrac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}), \\ \dfrac{d\mathbf{y}}{dt} = \mathbf{F}(\mathbf{x}, \mathbf{y}) - \mathbf{V}(\mathbf{y}), \end{cases}$$

where,

-$f(x)$ describes the dynamics of uncontaminated compartments when no contamination is present

-$F(x,y)$ contains the rates of new contamination into the exposed and infectious compartments,

-$V(y)$ represents transitions between infected compartments and exits due to recovery or decay.

Explicitly, the subsystem for uncontaminated compartments is given by

$$\frac{d}{dt}\begin{bmatrix} S \\ R \end{bmatrix} = \begin{bmatrix} P - \mu_S S \\ 0 \end{bmatrix}$$

which admits the globally asymptotically stable equilibrium

$$\mathbf{x}^* = \begin{bmatrix} S^* \\ R^* \end{bmatrix} = \begin{bmatrix} \frac{P}{\mu_S} \\ 0 \end{bmatrix}$$

The infected subsystem is

$$\begin{cases} \dfrac{dE}{dt} = \alpha SI - (\beta + \mu_E)E, \\ \dfrac{dI}{dt} = \beta E + PL - (\gamma + \delta + \mu_I)I. \end{cases}$$

At the contamination-free equilibrium where $y = 0$ and $S = S^*$, linearizing with respect to y yields the Jacobian matrix for infected compartments:

$$J_y = D\mathbf{F}(\mathbf{x}^*, 0) - D\mathbf{V}(0) = \begin{bmatrix} 0 & \alpha S^* \\ \beta & 0 \end{bmatrix} - \begin{bmatrix} \beta + \mu_E & 0 \\ 0 & \gamma + \delta + \mu_I \end{bmatrix} = \begin{bmatrix} -(\beta + \mu_E) & \alpha S^* \\ \beta & -(\gamma + \delta + \mu_I) \end{bmatrix}$$

Note that J_y is a Metzler matrix (off-diagonal entries non-negative) representing a cooperative and irreducible subsystem.

Using the conditions for Global Stability (Castillo-Chavez and Song Theorem)

i. ($H1$). The subsystem for x:

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x})$$

admits a globally asymptotically stable equilibrium x^* in the positively invariant set $\mathbb{R}^2_+$

In this case, the uninfected subsystem

$$\begin{cases} \dfrac{dS}{dt} = P - \mu_S S, \\ \dfrac{dR}{dt} = 0, \end{cases}$$

satisfies this, with equilibrium $x^* = (P/\mu_S, 0)$ globally attracting all positive initial data.

ii. ($H2$) The matrix J_y is a Metzler matrix and stable when $R_0 < 1$.

Here,

$$R_0 := \frac{\alpha\beta S^*}{(\beta + \mu_E)(\gamma + \delta + \mu_I)} < 1$$

ensures that the spectral radius of the next-generation matrix is less than 1, and thus all eigenvalues of J_y have negative real parts.

By the Castillo-Chavez and Song theorem, since the subsystem for uncontaminated compartments is globally asymptotically stable at x^*, and the infected subsystem at $y = 0$ is stable when $R_0 < 1$,

it follows that the Contamination-Free Equilibrium (CFE) of the complete SWAT-SEIR model is globally asymptotically stable whenever the basic reproduction number satisfies

$$\boxed{R_0 = \frac{\alpha\beta S^*}{(\beta + \mu_E)(\gamma + \delta + \mu_I)} < 1.}$$

This means that regardless of the initial contamination level, if $R_0 < 1$, the water system will eventually return to a contamination-free state.

F. Endemic Equilibrium (EE) of the SWAT-SEIR Model

The endemic equilibrium refers to a steady-state of the system where contamination persists at a positive, non-zero level. Formally, it is the equilibrium point

(S^*, E^*, I^*, R^*) with $E^* > 0$, $I^* > 0$, satisfying the system of algebraic equations obtained by setting the derivatives in the SWAT-SEIR model to zero:

$$\begin{cases} 0 = P - \alpha S^* I^* - \mu_S S^*, \\ 0 = \alpha S^* I^* - \beta E^* - \mu_E E^*, \\ 0 = \beta E^* + PL - \gamma I^* - \delta I^* - \mu_I I^*, \\ 0 = \gamma I^* + \delta I^* - \mu_R R^*. \end{cases} \tag{13}$$

(Step 1) Calculate Endemic Equilibrium, express R^* in terms of I^*. From the fourth equation of (13),

$$\mu_R R^* = (\gamma + \delta)\, I^*$$

$$\implies R^* = \frac{\gamma + \delta}{\mu_R} I^*.$$

(Step 2) Express I^* in terms of E^*. From the third equation of (13),

$$0 = BE^* + P\,L - (\gamma + \delta + \mu_I)\, I^*$$

which gives

$$I^* = \frac{\beta E^* + PL}{\gamma + \delta + \mu_I}.$$

(Step 3) Express E^* in terms of S^* and I^*. From the second equation of (13),

$$0 = \alpha\, S^* I^* - (\beta + \mu_E) E^*$$

which implies that,

$$E^* = \frac{\alpha S^* I^*}{\beta + \mu_E}.$$

(Step 4) Express S^* in terms of I^*. From the first equation,

$$0 = P - \alpha S^* I^* - \mu_S S^*,$$

rearranged as

$$\alpha S^* I^* + \mu_S S^* = P \Rightarrow S^* \alpha I^* + \mu_S) = P,$$

giving

$$S^* = \frac{P}{\alpha I^* + \mu_S}.$$

(Step 5) Substitute to find I^* Substitute E^* from Step 3 into Step 2:

$$I^* = \frac{\beta \frac{\alpha S^* I^*}{\beta + \mu_E} + PL}{\gamma + \delta + \mu_I}.$$

Multiply both sides by denominator:

$$(\gamma + \delta + \mu_I) I^* = \frac{\alpha \beta S^* I^*}{\beta + \mu_E} + PL$$

Rearranged as:

$$I^* \left[(\gamma + \delta + \mu_I) - \frac{\alpha \beta S^*}{\beta + \mu_E} \right] = PL.$$

Using the expression for S^* from Step (d):

$$I^* \left[(\gamma + \delta + \mu_I) - \frac{\alpha \beta}{\beta + \mu_E} \frac{P}{\alpha I^* + \mu_S} \right] = PL.$$

This nonlinear implicit equation in I^* can be solved numerically for $I^* > 0$ when it exists.

G. Biological Interpretation

i. The endemic equilibrium exists if $I^* > 0$, meaning persistent contamination in the water
system.

ii. Its existence requires that $R_0 > 1$, where the contamination transmission and progression overcome removal and recovery processes, allowing contamination to sustain.

iii. The model balances inflows (precipitation P, pollutant loading PL), transmission (α, β), and losses (μ_S, μ_E, μ_I, μ_R, γ, δ) at equilibrium.

iv. When $R_0 < 1$, $I^* = 0$ and the system tends towards the contamination-free equilibrium.

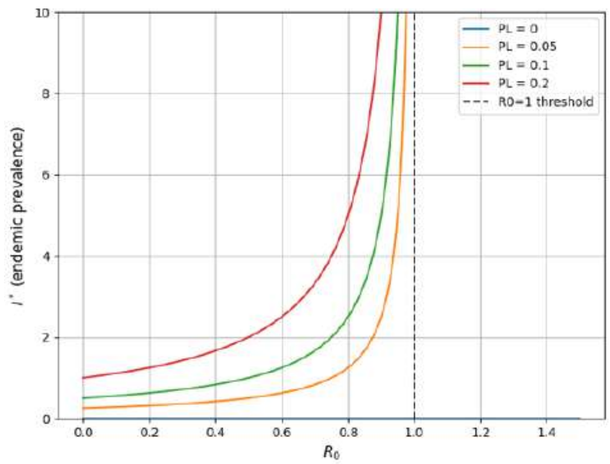


Figure 1: Endemic equilibrium I^* as a function of R_0

H. Local Stability of the Endemic Equilibrium

Let J denote the Jacobian matrix of system (1). It is given by

$$J = \begin{bmatrix} -\alpha I - \mu_S & 0 & -\alpha S & 0 \\ \alpha I & -(\beta + \mu_E) & \alpha S & 0 \\ 0 & \beta & -(\gamma + \delta + \mu_I) & 0 \\ 0 & 0 & (\gamma + \delta) & -\mu_R \end{bmatrix} \quad (14)$$

Evaluating at the endemic equilibrium E^*, we obtain $J(E^*)$. The characteristic polynomial is of the form

$$\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0,$$

where the coefficients ai are functions of the parameters and equilibrium values.

By the Routh–Hurwitz criterion, the endemic equilibrium E^* is locally asymptotically stable if

$$a_1 > 0, \quad a_2 > 0, \quad a_3 > 0, \quad a_4 > 0, \quad a_1 a_2 a_3 > a_3^2 + a_1^2 a_4.$$

I. Global Stability of the Endemic Equilibrium

To establish global stability, we construct the Lyapunov function

$$V(S,E,I,R) = (S - S^* - S^* \ln \tfrac{S}{S^*}) + (E - E^* - E^* \ln \tfrac{E}{E^*}) + (I - I^* - I^* \ln \tfrac{I}{I^*}) + (R - R^* - R^* \ln \tfrac{R}{R^*}).$$

Clearly, $V(S, E, I, R) \geq 0$ with equality if and only if

$$(S, E, I, R) = (S^*, E^*, I^*, R^*).$$

Differentiating along solutions of (1), we obtain

$$\frac{dV}{dt} = \left(1 - \frac{S^*}{S}\right)\frac{dS}{dt} + \left(1 - \frac{E^*}{E}\right)\frac{dE}{dt} + \left(1 - \frac{I^*}{I}\right)\frac{dI}{dt} + \left(1 - \frac{R^*}{R}\right)\frac{dR}{dt}.$$

After simplification, one shows that $\frac{dV}{dt} \leq 0$, with equality only at the endemic equilibrium.

Hence, by Lyapunov–LaSalle's invariance principle, the endemic equilibrium E^* is globally asymptotically stable.

III. SENSITIVITY ANALYSIS OF R_0

Recall the correct expression for the basic reproduction number R_0 in the SWAT-SEIR water contamination model:

$$R_0 = \frac{\alpha\beta P}{\mu_S(\beta+\mu_E)(\gamma+\delta+\mu_I)},$$

The normalized forward sensitivity index of R_0 with respect to a parameter x is defined as:

$$\Upsilon_x^{R_0} = \frac{\partial R_0}{\partial x} \times \frac{x}{R_0},$$

which quantifies the relative change in R0 given a relative change in x.

A. Calculation of Sensitivity Indices

i. Sensitivity with respect to α:

$$\frac{\partial R_0}{\partial \alpha} = \frac{\beta P}{\mu_S(\beta+\mu_E)(\gamma+\delta+\mu_I)} = \frac{R_0}{\alpha},$$

which gives

$$\Upsilon_\alpha^{R_0} = \frac{\partial R_0}{\partial \alpha}\frac{\alpha}{R_0} = 1.$$

ii. Sensitivity with respect to β: Express R_0 as

$$R_0 = \frac{\alpha P}{\mu_S(\gamma+\delta+\mu_I)} \times \frac{\beta}{\beta+\mu_E}.$$

Then,

$$\frac{\partial R_0}{\partial \beta} = \frac{\alpha P}{\mu_S(\gamma+\delta+\mu_I)} \times \frac{\mu_E}{(\beta+\mu_E)^2} = R_0 \times \frac{\mu_E}{\beta(\beta+\mu_E)},$$

and thus,

$$\Upsilon_\beta^{R_0} = \frac{\partial R_0}{\partial \beta} \cdot \frac{\beta}{R_0} = \frac{\mu_E}{\beta+\mu_E}.$$

iii. Sensitivity with respect to P:

$$\frac{\partial R_0}{\partial P} = \frac{\alpha\beta}{\mu_S(\beta+\mu_E)(\gamma+\delta+\mu_I)} = \frac{R_0}{P},$$

so

$$\Upsilon_P^{R_0} = 1.$$

iv. Sensitivity with respect to μ_S : From the denominator,

$$\frac{\partial R_0}{\partial \mu_S} = -\frac{R_0}{\mu_S},$$

leading to

$$\Upsilon_{\mu_S}^{R_0} = -1.$$

v. Sensitivity with respect to μ_E :

$$R_0 = \frac{\alpha\beta P}{\mu_S(\gamma+\delta+\mu_I)} \cdot \frac{1}{\beta+\mu_E}.$$

Differentiating,

$$\frac{\partial R_0}{\partial \mu_E} = -\frac{\alpha\beta P}{\mu_S(\beta+\mu_E)^2(\gamma+\delta+\mu_I)} = -\frac{R_0}{\beta+\mu_E},$$

and

$$\Upsilon_{\mu_E}^{R_0} = -\frac{\mu_E}{\beta+\mu_E}.$$

vi. Sensitivity with respect to γ, δ, μ_I :

Let

$$M = \gamma + \delta + \mu_I ,$$

then

$$R_0 = \frac{\alpha\beta P}{\mu_S(\beta+\mu_E)M}.$$

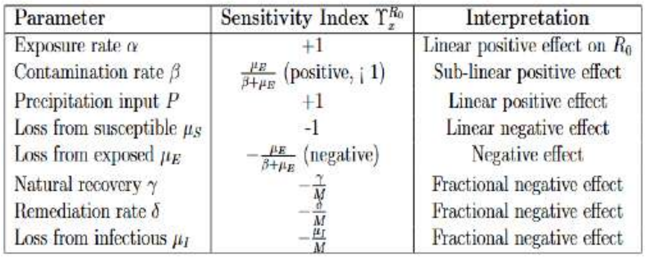
Derivative w.r.t. γ (similarly for δ, μ_I):

$$\frac{\partial R_0}{\partial \gamma} = -\frac{R_0}{M},$$

giving

$$\Upsilon_\gamma^{R_0} = -\frac{\gamma}{M}, \quad \Upsilon_\delta^{R_0} = -\frac{\delta}{M}, \quad \Upsilon_{\mu_I}^{R_0} = -\frac{\mu_I}{M}.$$

B. Summary of Sensitivity Indices



Parameter	Sensitivity Index $\Upsilon_x^{R_0}$	Interpretation
Exposure rate α	+1	Linear positive effect on R_0
Contamination rate β	$\frac{\mu_E}{\beta+\mu_E}$ (positive, ¡ 1)	Sub-linear positive effect
Precipitation input P	+1	Linear positive effect
Loss from susceptible μ_S	-1	Linear negative effect
Loss from exposed μ_E	$-\frac{\mu_E}{\beta+\mu_E}$ (negative)	Negative effect
Natural recovery γ	$-\frac{\gamma}{M}$	Fractional negative effect
Remediation rate δ	$-\frac{\delta}{M}$	Fractional negative effect
Loss from infectious μ_I	$-\frac{\mu_I}{M}$	Fractional negative effect

Table 1: Normalized Forward Sensitivity Indices of R_0

C. Interpretation

Parameters in numerator (α,β,P) positively influence R_0, increasing contamination risk when raised. Parameters related to removal or loss (μ_S, μ_E, μ_I, γ, δ) negatively influence R_0, helping reduce contamination potential. Sensitivity indices support targeting the most influential parameters to effectively control contamination spread.

IV. FOURTH-ORDER RUNGE-KUTTA METHOD

The Fourth-Order Runge-Kutta (RK4) method is a powerful numerical technique used to approximate solutions of ordinary differential equations (ODEs), especially systems of ODEs.

Given a system of ODEs in the form:

$$\frac{dX}{dt} = F(X,t)$$

where $X = (S,E,I,R)^T$ is the vector of dependent variables, and $F(X,t)$ is the vector of functions representing the right-hand sides of your ODEs.

For your specific model, $X = (S,E,I,R)^T$, and the function $F(X,t)$ is:

$$F(X,t) = F(S,E,I,R) = \begin{pmatrix} P - \alpha SI - \mu_S S \\ \alpha SI - \beta E - \mu_E E \\ \beta E + PL - \gamma I - \delta I - \mu_I I \\ \gamma I + \delta I - \mu_R R \end{pmatrix}$$

The RK4 method approximates the solution by calculating four "slopes" (or intermediate steps) within each time interval h, and then uses a weighted average of these slopes to determine the next value of X.

Let $X_n = (S_n, E_n, I_n, R_n)$ be the values of the variables at time t_n. We want to find X_{n+1} at time $t_{n+1} = t_n + h$. Here's how to apply RK4 for one-time step:

The General RK4 Formula is given by:

$$X_{n+1} = X_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where:

$$\begin{aligned} k_1 &= hF(X_n, t_n) \\ k_2 &= hF\left(X_n + \frac{1}{2}k_1, t_n + \frac{1}{2}h\right) \\ k_3 &= hF\left(X_n + \frac{1}{2}k_2, t_n + \frac{1}{2}h\right) \\ k_4 &= hF(X_n + k_3, t_n + h) \end{aligned}$$

Applying it to your system (S, E, I, R):

Let's denote F_S, F_E, F_I, F_R as the right-hand side functions for each differential equation:

$$\begin{aligned} F_S(S,E,I,R) &= P - \alpha SI - \mu_S S \\ F_E(S,E,I,R) &= \alpha SI - \beta E - \mu_E E \\ F_I(S,E,I,R) &= \beta E + PL - (\gamma + \delta + \mu_I)I \\ F_R(S,E,I,R) &= (\gamma + \delta)I - \mu_R R \end{aligned}$$

And let $X_n = (S_n, E_n, I_n, R_n)$.

Step 1: Calculate k_1

This is the slope at the beginning of the interval.

$$\begin{aligned} k_{1,S} &= h \cdot F_S(S_n, E_n, I_n, R_n) \\ k_{1,E} &= h \cdot F_E(S_n, E_n, I_n, R_n) \\ k_{1,I} &= h \cdot F_I(S_n, E_n, I_n, R_n) \\ k_{1,R} &= h \cdot F_R(S_n, E_n, I_n, R_n) \end{aligned}$$

Step 2: Calculate k_2

This is the slope at the midpoint, using X_n and k_1.

$$\begin{aligned} S_{n,k2} &= S_n + \frac{1}{2}k_{1,S} \\ E_{n,k2} &= E_n + \frac{1}{2}k_{1,E} \\ I_{n,k2} &= I_n + \frac{1}{2}k_{1,I} \\ R_{n,k2} &= R_n + \frac{1}{2}k_{1,R} \end{aligned}$$

Then,

$$\begin{aligned} k_{2,S} &= h \cdot F_S(S_{n,k2}, E_{n,k2}, I_{n,k2}, R_{n,k2}) \\ k_{2,E} &= h \cdot F_E(S_{n,k2}, E_{n,k2}, I_{n,k2}, R_{n,k2}) \\ k_{2,I} &= h \cdot F_I(S_{n,k2}, E_{n,k2}, I_{n,k2}, R_{n,k2}) \\ k_{2,R} &= h \cdot F_R(S_{n,k2}, E_{n,k2}, I_{n,k2}, R_{n,k2}) \end{aligned}$$

Step 3: Calculate k_3

This is also the slope at the midpoint, but using X_n and k_2 (a refined estimate).

$$\begin{aligned} S_{n,k3} &= S_n + \frac{1}{2}k_{2,S} \\ E_{n,k3} &= E_n + \frac{1}{2}k_{2,E} \\ I_{n,k3} &= I_n + \frac{1}{2}k_{2,I} \\ R_{n,k3} &= R_n + \frac{1}{2}k_{2,R} \end{aligned}$$

Then,

$$\begin{aligned} k_{3,S} &= h \cdot F_S(S_{n,k3}, E_{n,k3}, I_{n,k3}, R_{n,k3}) \\ k_{3,E} &= h \cdot F_E(S_{n,k3}, E_{n,k3}, I_{n,k3}, R_{n,k3}) \\ k_{3,I} &= h \cdot F_I(S_{n,k3}, E_{n,k3}, I_{n,k3}, R_{n,k3}) \\ k_{3,R} &= h \cdot F_R(S_{n,k3}, E_{n,k3}, I_{n,k3}, R_{n,k3}) \end{aligned}$$

Step 4: Calculate k_4

This is the slope at the end of the interval, using X_n and k_3.

$$\begin{aligned} S_{n,k4} &= S_n + k_{3,S} \\ E_{n,k4} &= E_n + k_{3,E} \\ I_{n,k4} &= I_n + k_{3,I} \\ R_{n,k4} &= R_n + k_{3,R} \end{aligned}$$

Then,

$$\begin{aligned} k_{4,S} &= h \cdot F_S(S_{n,k4}, E_{n,k4}, I_{n,k4}, R_{n,k4}) \\ k_{4,E} &= h \cdot F_E(S_{n,k4}, E_{n,k4}, I_{n,k4}, R_{n,k4}) \\ k_{4,I} &= h \cdot F_I(S_{n,k4}, E_{n,k4}, I_{n,k4}, R_{n,k4}) \\ k_{4,R} &= h \cdot F_R(S_{n,k4}, E_{n,k4}, I_{n,k4}, R_{n,k4}) \end{aligned}$$

Step 5: Compute X_{n+1}

Finally, update the values for the next time step:

$$\begin{aligned} S_{n+1} &= S_n + \frac{1}{6}(k_{1,S} + 2k_{2,S} + 2k_{3,S} + k_{4,S}) \\ E_{n+1} &= E_n + \frac{1}{6}(k_{1,E} + 2k_{2,E} + 2k_{3,E} + k_{4,E}) \\ I_{n+1} &= I_n + \frac{1}{6}(k_{1,I} + 2k_{2,I} + 2k_{3,I} + k_{4,I}) \\ R_{n+1} &= R_n + \frac{1}{6}(k_{1,R} + 2k_{2,R} + 2k_{3,R} + k_{4,R}) \end{aligned}$$

Process Flow:

1. Initialize: Set the initial conditions (S_0, E_0, I_0, R_0) at t_0.
2. Choose Time Step: Select a suitable step size h. The RK4 method is known for its accuracy, but a smaller h will generally lead to more accurate results at the cost of more computational time. The paper suggests $h = 0.1$ or $h = 1$ are stable.
3. Iterate: Repeat Steps 1-5 for each time step from t_0 up to the desired simulation end time. Store (S_n, E_n, I_n, R_n) at each t_n to generate the time series of the solution.

Conceptual process:

To simulate for 100 days with $h = 1$ day, we start with S_0, E_0, I_0, R_0 at $t = 0$ and use RK4 calculations to find S_1, E_1, I_1, R_1 at $t = 1$. Then we use S_1, E_1, I_1, R_1 as the input for the next step to find S_2, E_2, I_2, R_2 at $t = 2$. We continue this process for 100 iterations.

Why we use RK4:

Accuracy: It's a fourth-order method, meaning its local truncation error is of order $O(h^5)$, and its global truncation error is of order $O(h^4)$. This makes it quite accurate for many problems.

Stability: It is generally stable for many types of ODEs, and as the paper mentions, it's conditionally stable, meaning the choice of h is important.

Relatively Simple to Implement: While it involves four slopes, the pattern is straightforward, making it amenable to programming.

CONCLUSION

This SWAT-SEIR model represents the dynamics of water contamination influenced by climatic factors, establishing that contamination-free equilibrium is locally and globally stable when the basic reproduction number $R_0<1$, while contamination persists at an endemic equilibrium when $R_0>1$. The analytical results for the model, supported by sensitivity analysis and numerical simulations in the RK4 method, provide critical insights into controlling contamination spread and highlight key parameters for effective intervention.

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