(Knowledge for Development)

KIBABII UNIVERSITY

UNIVERSITY EXAMINATIONS

2024/2025 ACADEMIC YEAR

FOURTH YEAR SECOND SEMESTER

MAIN EXAMINATION

FOR THE DEGREE OF BACHELOR OF EDUCATION AND BACHELOR OF SCIENCE

COURSE CODE: MAP 427/MAT406

COURSE TITLE: FIELD THEORY

DATE: 22/04/2025 **TIME:** 2:00 PM – 4:00 PM

INSTRUCTIONS TO CANDIDATES

Answer Question One and Any other TWO Questions

TIME: 2 Hours

QUESTION ONE (30 MARKS)

a) Define the following terms
   i. Division Ring (2marks)
   ii. Division algorithm (1mark)
   iii. Homomorphism of Rings (2marks)

b) What is the splitting field for $f(x) = x^4 + 4$ over $\mathbb{Q}$ (5marks)

c) Use division algorithm to divide $g(x) = 5x^4 + 3x^3 + 1$ by $f(x) = 3x^2 + 2x + 1$ in $\mathbb{Z}_7[x]$ (6marks)

d) Prove that $f(x)$ is reducible over a field F iff it has at least one zero in F (6marks)

e) If $p(x)$ is an irreducible polynomial over a field $\mathbb{F}$, prove that there exists an extension field of F that contains a zero of $p(x)$ (8marks)

QUESTION TWO (20 MARKS)

a) Define a Finitely generated field extension (2marks)

b) Consider $p(x) = x^2 + 1$ which is irreducible over $\mathbb{Z}_2$. Determine all the elements of the field $\mathbb{Z}_2[x]/p(x)$ and construct their multiplication and addition tables. Is $\mathbb{Z}_2[x]/p(x)$ a field? Explain (10marks)

c) Find the product and the $deg(f(x).g(x))$ of $f(x) = 2x^3 + x + 1$ and $g(x) = 2x^3 + 3x^2 + 4$ (8marks)

QUESTION THREE (20 MARKS)

a) State the Einstein criterion (3marks)

b) Determine whether $f(x) = 2x^6 - 3x^4 + 6x^2 - 6x + 12$ is irreducible or not using the Einstein criterion (3marks)

c) Prove that the characteristic of a field $\mathbb{F}$ is either zero or a prime p (6marks)

d) Express $x^4 - 9$ as a product of irreducible factors in $\mathbb{Q}[x]$ and $\mathbb{C}[x]$ (4marks)

e) If R is an integral domain and $p(x)$ and $q(x)$ are non-zero elements of $R[x]$, prove that $\deg(p(x).q(x)) = deq\ p(x) + \deg q(x)$ (4marks)

QUESTION FOUR (20 MARKS)

a) Define a Minimal polynomial (2marks)

b) Find the minimal polynomial of $\alpha = \sqrt{3 + \sqrt{7}}$ over the field $\mathbb{Q}$ and state the degree of α (4marks)

c) Prove that given a field $\mathbb{F}$, $\mathbb{F}(\alpha, \beta) = (\mathbb{F}(\alpha))(\beta)$ (4marks) d)

i. Define the GcD of polynomials (3marks)

ii. Find the gcd of $f(z) = z^4 + 4z^3 + 5z^2 + 4$ and $g(z) = z^2 + 5z + 4$ in $\mathbb{Z}_7(z)$ (7marks)

QUESTION FIVE (20 MARKS)

a) Let F be a field, show that $f(x)$ is a unit in $F[x]$ iff $f(x)$ is a non-zero constant polynomial (5marks)

b) Prove that $\mathbb{Z}_5$ is an integral domain (5marks)

c) Determine whether the polynomial $f(x) = 2x^2 + x + 3$ is reducible or not in $\mathbb{Z}_5$ (5marks)

d) Prove that for any commutative ring R with unity, the ring $R[x]$ of polynomials over R contains a subring R' that is isomorphic to R (5marks)