KIBABII UNIVERSITY

UNIVERSITY EXAMINATIONS

2022/2023 ACADEMIC YEAR

SECOND YEAR SECOND SEMESTER FOR THE DEGREE OF

BACHELOR OF EDUCATION SCIENCE

COURSE CODE: **SPH 222**

COURSE TITLE: **QUANTUM MECHANICS**

DATE:17/04/2023 **TIME: 9:00-11:00AM**

INSTRUCTIONS TO CANDIDATES

Answer question ONE and any TWO of the remaining
Symbols used bear the usual meaning.

KIBU observes ZERO tolerance to examination cheating

You may find the following information useful:-

Speed of light in the vacuum, c = 3.00×10^8 — ms^{-1} ; Planck's constant, h = 6.63×10^{-34} J s

$\hbar = \frac{h}{2\pi} = 1.055 \times 10^{-34} J.s$;Mass of a proton $m_p = 1.66 \times 10^{-27}$ kg

Mass of the electron, $m_e = 9.11 \times 10^{-31}$ kg

1 eV = 1.6×10^{-19} J ; 1MeV = 10^6 eV ; 1nm = 10^{-9} m.

$$\int \sin^2 bx\, dx = \frac{x}{2} - \frac{1}{4b}\sin(2bx)$$

$$\int_{-\infty}^{+\infty} e^{-ax^2}\, dx = \sqrt{\frac{\pi}{a}}$$

$$\int_{-\infty}^{+\infty} x.e^{-2ax^2}\, dx = 0$$

$$\int x^{2n} e^{-bx^2}\, dx = \frac{1\times 3\times ....\times(2n-1)}{2^{n+1}}\left(\frac{\pi}{b^{2n+1}}\right)^{\frac{1}{2}} \quad ; \quad n = 1, 2, 3, ...$$

$$\sin^2 A = \frac{1}{2}[1 - \cos 2A]$$

QUESTION ONE [30 MARKS]

i. Write down the **time-dependent** and **time-independent** Schrodinger equation for the wave function $\Psi(x, t)$ of a particle of mass m moving in one dimension x in the potential V (x). [4 Marks]

ii. Write down the **normalization condition** for the wave function $\Psi(x)$ of a particle which is restricted to move in the interval $-\infty \leq x \leq +\infty$. [2Marks]

iii. A particle is constrained to move in the one-dimensional interval $-a \leq x \leq a$. Write down the definition of the **expectation value** $\langle x \rangle$ and $\langle x^2 \rangle$ of the coordinate x when the wave function of the particle is $\psi(x)$. [3 Marks]

iv. What is the quantum **mechanical interpretation** of ψ and $\psi\psi^*$, where ψ is a solution of the Schrödinger equation? Why does ψ have to be square-integrable? What does this mean in mathematical terms?

v. Find the normalization constant, N, for the wave function, [3 Marks]

$$\psi(x,t) = Ne^{-\frac{ax^2}{2}} e^{-\frac{i\omega t}{\hbar}}$$

For $-\infty \leq x \leq +\infty$. in the interval

Write down the normalized wave function. Use the standard integral

$$\int_{-\infty}^{+\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$$

vi. A one-particle, one-dimensional system has a normalized wavefunctions of the for

$$\psi_n(x) = \sqrt{\frac{2}{L}} . \sin\left(\frac{n\pi x}{L}\right)$$

at t = 0, where L = 1 nm. At t= 0, the particle's position is measured. Find the probability that the measured value lies between x = 0.1 nm and x = 0.2 nm. [2 Marks]

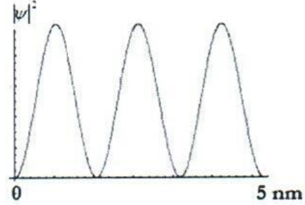
vii. Consider a particle in the ground state of an infinite potential box of length L. The wave functions of the particle in an infinite potential well are given by

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

a. Find the probability density $|\Psi|^2$ for the ground state. [1 Marks]

b. What is the probability of finding a particle in the interval between x = 0.50L and x = 0.51L (in the ground state) ? [2 Marks]

viii. Consider an electron trapped in a 1D well with L = 5 nm. Suppose the electron is in the following state:



Assume that the potential seen by the electron is approximately that of an infinite square well. What is the energy of the electron in this state (in eV)? [3 marks]

ix. Explain what is meant by the orthogonality of two wavefunctions $\psi_1(x)$ and $\psi_2(x)$, in the quantum mechanics of a particle on a line $-\infty < x < \infty$ [3 Marks]

x. An electron is acted by a potential V(x) within the region . Its wavefunction is given as

$$\psi(x,t) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right) e^{-i\omega t} \qquad 0 \le x \le a$$

Use the **time dependent** Schrodinger equation to show that the potential V(x) acting on
the electron is given by [4 marks]

$$V = \frac{-\hbar^2\pi^2}{2ma^2} + \hbar\omega$$

xi. For a 1-D system momentum operator $\hat{p}_x = -i\hbar\frac{d}{dx}$ and the position operator $\hat{X} = x$ *show that* $\left[\hat{p}_x\,,\,\hat{X}\right] = -i\hbar$.

[3 marks]

QUESTION TWO [20 MARKS]

a) Express the integral $\int \psi_n^*(x)\psi_m(x)dx = \delta_{nm}$ in terms of the Dirac's bra and ket notation [2 marks]

b) Write down the time independent eigen-value Schrodinger equation [2 marks]

c) A quantum system has a set of eigenstates $u_n(x)$, with energies E_n. The system is placed in a state ψ that is not an eigenstate; use the fact that the u_n are a complete set to show that the expectation value of the Hamilitonian, $\langle\psi|H|\psi\rangle$, always overestimates the ground-state energy. [10 marks]

d) A region of space has a potential step such that particles have a wave function given by

$$\psi(x,t) = \begin{cases} \frac{5a}{\sqrt{2}} e^{i(K_1 x - Et/\hbar)} + \frac{3a}{\sqrt{2}} e^{i(-K_1 x - Et/\hbar)}, & x < 0 \\ \frac{8a}{\sqrt{2}} e^{i(K_2 x - Et/\hbar)}, & x > 0 \end{cases}$$

The incident particles, initially at $x << 0$, are initially travelling in the positive x direction.

(i) What fraction of the incident particles will be reflected ? [3marks]

(ii) What is K_2 / K_1 ? [3 marks]

QUESTION THREE [20 MARKS]

Consider a particle of mass m inside a box of size L with infinite walls,

$$V(x) = \begin{cases} 0 & \text{for } 0 \leq x \leq L \\ \infty & \text{elsewhere} \end{cases}$$

The wave function is specified at $t = 0$ to be

$$\psi(x,t=0) = C\left[3\sin\left(\frac{2\pi x}{L}\right) - 2\sin\left(\frac{3\pi x}{L}\right)\right]$$

a) Show that the above expression for □$(x,t = 0)$ can also be written as [3marks]

$$\psi(x,t=0) = C\sqrt{\frac{L}{2}}\left[3u_2(x) - 2u_3(x)\right],$$

given that the eigen-functions for a wave function in an infinite potential well is given by

$$u_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{2\pi x}{L}\right)$$

b) Determine the normalization coefficient C. [5 marks]

c) Expand the wave function at the initial time $\psi(x,t=0)$ in terms of eigenfunctions of the infinite box, i.e. determine the expansion coefficients c_n [3 marks]

d) Write down $\psi(x,t)$, at an arbitrary later time t. [5 marks]

e) If a measurement of the particle's energy at time t is performed, what will be the possible outcomes, and with what probability will those values be measured? What is the average energy $\langle E \rangle$ of the particle in the box? Is $\langle E \rangle$ changed by the measurement? [4 marks]

QUESTION FOUR [20 MARKS]

a) Use the normalization integral to show that the normalization constant N=(2/L)$^{1/2}$ and hence write down the normalized wave function for

$$\psi(x,t) = Ne^{-\frac{ax^2}{2}} e^{-\frac{i\varepsilon_0 t}{\hbar}},$$

in the same interval $-\infty \le x \le +\infty$ [4 marks]

b) Determine the expectation value $\langle x \rangle$ and $\langle x^2 \rangle$ and hence show that the uncertainty

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \frac{1}{\sqrt{2a}}$$

for the wave function described in (a) above [9 marks]

c) Determine the expectation value $\langle p \rangle$ and $\langle p^2 \rangle$ and hence show that the Uncertainty in momentum, $\Delta p = \sqrt{\langle P^2 \rangle - \langle P \rangle^2} = \hbar\sqrt{\frac{a}{2}}$ [5 marks]

d) Hence demonstrate that : $\Delta p\, \Delta x = \frac{\hbar}{2}$ and comment very briefly how this compares to the 2 uncertainty principle [2 marks]

QUESTION FIVE [20 MARKS]

a) Write down the time-independent Schrodinger equation for a particle in a one-dimensional harmonic oscillator potential, $V = \frac{m\omega^2 x^2}{2}$ [3 marks]

b) The ground-state wave function is of the form $\psi = A\exp(-\alpha x^2)$. Determine the constant□, and hence the ground-state energy. [10 marks]

c) A particle of mass m is confined to a harmonic oscillator potential given by $V = m\omega^2 x^2 / 2$, where $\omega = K/m$ and K is the force constant. The particle is in a state described by the wave function

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$$\psi(x,t) = A e^{\left(\frac{-m x^2 \omega}{2\hbar} - \frac{i\omega t}{2}\right)}$$

Verify that this is a solution of Schrodinger's equation. [7 marks]