INTEGRAL FUNCTIONALS, NORMAL INTEGRANDS AND MEASURABLE SELECTIONS

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A fundamental notion in many areas of mathematics, including optimization, probability, variational problems, functional analysis and operator theory, is that of an integral functional. By this is meant an expression of the form

$$I_f(x) = \int_S f(s,x(s))\mu(ds), \qquad x \in X,$$

where X is a linear space of measurable functions defined on a measure space $(S,\mathcal{A},\mu)$ and having values in a linear space E. The function $f\colon S \times E \to \bar{R}$ is the associated <u>integrand</u>.

Classically, only finite integrands on $S \times R^n$ were studied, usually under the assumption that $f(s,x)$ was continuous in x and measurable in s (the Carathéodory condition). However, from the modern point of view it is essential to admit possibly infinite values for f and I_f, since it is in this way that important kinds of constraints can most efficiently be represented. Such integrands require a distinctly new theoretical approach, where questions of measurability and the existence of measurable selections are prominent and are reflected in a concept of "normality".

The purpose of these notes is to provide a relatively thorough treatment of the most common case in applications, that where $E = R^n$. While many of the results have extensions in one way or another beyond this case, as indicated to some extent in the text, these are often more complicated technically and may require further restrictions. For example, it is only for R^n that one presently knows how to develop a complete theory without assuming that the measurable space is complete, an assumption which appears to be awkward in some situations. In treating infinite-dimensional spaces E, there are the usual problems of the multiplicity of topologies and dualities which must be ironed out. It is desirable, therefore, to have available a full and consistent exposition of the details in the basic case of $E = R^n$, freeing one from the need to search for auxilliary results through sequences of papers with varying frameworks.

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The material below is divided in three principle sections. First we present the theory of measurable closed-valued multifunctions. Equivalent properties, any of which could actually be used as the definition of measurability, are discussed, and the basic measurable selection theorem of Kuratowski and Ryll-Nardzewski is derived via a stronger theorem on the existence of Castaing representations. (The proof, which is given in full, is simpler for R^n than in the more general case usually seen in the literature.) Much effort is devoted to establishing convenient means of verifying that a multifunction is indeed measurable.

The second part applies the results on measurable multifunctions to the study of normal integrands, a concept originally introduced by the author [1] in a setting of convexity, but developed here in more general terms. Again the emphasis is on measurability questions and the manufacture of tools which make easier the verification of "normality". Normal integrands are also important in the generation of measurable multifunctions given by systems of constraints, subdifferential mappings etc.

These technical developments come to fruition in the theory of integral functionals presented in the third section of the notes. It is here also that convex analysis comes more to the front of the stage. This is due to natural considerations of duality, which are always important in a setting of functional analysis, as well as deeper reasons related to Liapunov's theorem and involving the weak compactness of leve sets of integral functionals.

For obvious reasons of space, the discussion is limited to integral functionals on decomposable function spaces, such as Lebesgue spaces. These are characterized by the validity of a fundamental result on the interchange of integration and minimization. The treatment of more general function spaces usually relies heavily on this, more basic theory, as for example the case of Banach spaces of continuous functions as developed in [2], or the spaces of differentiable functions encountered in variational problems (cf. [13], [15], [26], [32]). We have made no attempt to cover the many results in such directions.