Dedicated to Professor Gheorghe Bucur
on the occasion of his 70th birthday

ON THE FRACTIONAL CAUCHY PROBLEM ASSOCIATED WITH A FELLER SEMIGROUP

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Starting from the usual Cauchy problem, we give a pseudodifferential representation for the solution of the fractional Cauchy problem associated with a Feller semigroup.

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1. INTRODUCTION

Fractional derivatives are used to model anomalous diffusion, which occurs when the particles spread in a different manner than the prediction of the classical diffusion equation

$$\frac{\partial}{\partial t}u(x,t) = D\frac{\partial^2}{\partial x^2}u(x,t), \quad u(x,0) = f(x).$$

The solution $u\,(x,t)$ depends on location $x \in \mathbf{R}$ and time $t \geq 0$ and models the dispersion. A known model for an anomalous diffusion (see [6]) is the fractional diffusion equation, where the usual second derivative in space is replaced by a fractional derivative of order α, $0 < \alpha < 2$,

$$\frac{\partial}{\partial t}u(x,t) = D\frac{\partial^\alpha}{\partial x^\alpha}u(x,t), \quad u(x,0) = f(x).$$

We observe that $\frac{\partial^\alpha}{\partial x^\alpha}$ is a pseudodifferential operator. Thus, we can extend this equation to

$$\frac{\partial u}{\partial t}(x,t) = (Au(\cdot\,,t))(x), \quad u(x,0) = u_0(x),$$

where A is a pseudodifferential operator. A study of the solutions of a generalized reaction-diffusion equation of the form

$$\frac{\partial u}{\partial t}\,(x,t) = \left(Au\,(\cdot\,,t)\right)(x) + f\,(x,u\,(x,t))\,, \quad u\,(x,0) = u_0\,(x)\,,$$

where A is a pseudodifferential operator which generates a Feller semigroup, was given in [9].

In this paper we consider the fractional Cauchy problem

$$\frac{\partial^\beta}{\partial t^\beta}u(x,t) = (Au(\cdot,t))(x), \quad u(x,0) = f(x),$$

where $\frac{\partial^\beta}{\partial t^\beta}u\,(x,t)$ is the Caputo fractional derivative in time and A is a pseudodifferential operator which generates a Feller semigroup. In [1] and [2] was shown that the solution of fractional Cauchy problem, where $0 < \beta < 1$, $t \geq 0$ and A is the generator of bounded continuous semigroup $\{T\,(t)\}_{t\geq 0}$ on the Banach space X, can be expressed as an integral transform of the solution to the initial Cauchy problem

$$\frac{\partial}{\partial t}u(x,t) = (Au(\cdot,t))(x), \quad u(x,0) = f(x).$$

Starting from this integral transform, we give a formula for the solution $u(x,t) = S(t)f(x)$ of the fractional Cauchy problem. We show that $\{S(t)\}_{t\geq 0}$ is a family of pseudodifferential operators. Their symbols are obtained by transformation of the symbols of the semigroup $\{T(t)\}_{t\geq 0}$, where $u(x,t) = T(t)f(x)$ is the solution to the initial Cauchy problem.

2. INTEGRAL REPRESENTATION OF THE OPERATORS WHICH FORM A FELLER SEMIGROUP

Let $(X, \|\cdot\|)$ be a Banach space. $\{T(t)\}_{t\geq 0}$ is a strongly continuous semigroup on X if for any $t \geq 0$, $T(t) : X \to X$ is a linear operator and there exists $M > 0$ such that $\|T(t)x\| \leq M\|x\|$, $T(0) = I$, $T(t+s) = T(t)T(s)$ for $t, s \geq 0$, and $t \to T(t)x$ is continuous in the norm $\|\cdot\|$, for all $x \in X$. The generator $(A, D(A))$ of the semigroup $\{T(t)\}_{t\geq 0}$, is defined by

$$D(A) = \left\{x \in X \mid \lim_{t\to 0+} \frac{T(t)x - x}{t} \text{ exists}\right\}, \quad Ax = \lim_{t\to 0+} \frac{T(t)x - x}{t},$$

where we suppose that the limit exists for at least some nonzero $x \in X$. $u\,(t) = T\,(t)\,f$ solves the abstract Cauchy problem

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) = Au(t), \quad u(0) = f,$$

for $f \in D(A)$.

In the following, we denote by $C_\infty(\mathbf{R}^n)$ the Banach space of all continuous functions on $\mathbf{R}^n$ vanishing at infinity with the supremum norm $\|\cdot\|_\infty$ and by $C_0^\infty(\mathbf{R}^n)$ the set of all C^∞-functions on $\mathbf{R}^n$ with compact support. $S(\mathbf{R}^n)$

will be the Schwartz space, i.e., the set of all functions $\varphi \in C^\infty(\mathbf{R}^n)$ such that $\sup_{x\in\mathbf{R}^n} |x^\beta \partial^\alpha \varphi(x)| < \infty$ for all multi-indices α and β. $S(\mathbf{R}^n)$ is dense in $C_\infty(\mathbf{R}^n)$.

A function $a : \mathbf{R}^n \to \mathbf{C}$ is called *negative definite* if $a(0) \geq 0$ and $\xi \to \mathrm{e}^{-ta(\xi)}$ is a positive definite function for all $t > 0$. A continuous negative definite function a is described by *Lévy-Khinchin formula*

$$a(\xi) = c + \mathrm{i}b\cdot\xi + q(\xi) + \int_{\mathbf{R}^n\setminus\{0\}} \left[1 - \mathrm{e}^{-\mathrm{i}\xi\cdot y} - \mathrm{i}\frac{\xi\cdot y}{1+|y|^2}\right] \frac{1+|y|^2}{|y|^2}\mathrm{d}\mu(y)$$

with $c \geq 0$, $b \in \mathbf{R}^n$, q a continuous non-negative definite quadratic form on $\mathbf{R}^n$ and μ, called *measure Lévy*, a σ-finite Borel measure on $\mathbf{R}^n \setminus \{0\}$ such that

$$\int_{\mathbf{R}^n\setminus\{0\}} \min(1, |y|^2)\mathrm{d}\mu(y) < \infty.$$

The general form of a *pseudodifferential operator* is

$$p(x, D)\varphi(x) = (2\pi)^{-(n/2)} \int_{\mathbf{R}^n} \mathrm{e}^{\mathrm{i}x\cdot\xi} p(x,\xi)\widehat{\varphi}(\xi)\mathrm{d}\xi,$$

for $\varphi \in C_0^\infty(\mathbf{R}^n)$, where

$$\widehat{\varphi}(\xi) = (2\pi)^{-(n/2)} \int_{\mathbf{R}^n} \mathrm{e}^{-\mathrm{i}x\cdot\xi}\varphi(x)\mathrm{d}x$$

is the Fourier transform. $p(x, \xi)$ is called the *symbol* of the operator $p(x, D)$ (see, for example, [4]).

Let $A : D(A) \to C_\infty(\mathbf{R}^n)$ be a linear operator, where $D(A)$ is a linear dense subspace of $C_\infty(\mathbf{R}^n)$. A satisfies *the positive maximum principle* on $D(A)$ if for all $u \in D(A)$ and $x_0 \in \mathbf{R}^n$ such that

$$\sup_{x\in\mathbf{R}^n} u(x) = u(x_0) \geq 0$$

it follows that $Au(x_0) \leq 0$.

THEOREM 2.1 ([3]). *Let $A : C_0^\infty(\mathbf{R}^n) \to C_b(\mathbf{R}^n)$ be a linear operator satisfying the positive maximum principle. Then*

$$Au(x) = -(2\pi)^{-(n/2)} \int_{\mathbf{R}^n} \mathrm{e}^{\mathrm{i}x\cdot\xi} a(x,\xi)\widehat{u}(\xi)\mathrm{d}\xi,$$

where $a : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{C}$ is a locally bounded function such that for any fixed $x \in \mathbf{R}^n$, $\xi \to a(x, \xi)$ is a continuous negative definite function.

The *convolution semigroup* on $C_\infty(\mathbf{R}^n)$ generated by a is defined by the formula

$$T(t)u(x) = (2\pi)^{-(n/2)} \int_{\mathbf{R}^n} \mathrm{e}^{\mathrm{i}x\cdot\xi} p_t(\xi)\widehat{u}(\xi)\mathrm{d}\xi,$$

for each $t > 0$ and $u \in S(\mathbf{R}^n)$, where $p_t(\xi) = \mathrm{e}^{-ta(\xi)}$. In this case, we observe that for any $t > 0$ the symbol is p_t (note that there is no x-dependence). The function $\xi \to p_t(\xi)$ is a positive definite function and the infinitesimal generator of $\{T(t)\}_{t\geq 0}$ is

$$Au(x) = -(2\pi)^{-(n/2)} \int_{\mathbf{R}^n} \mathrm{e}^{\mathrm{i}x\cdot\xi} a(\xi)\widehat{u}(\xi)\mathrm{d}\xi,$$

for all $u \in C_0^\infty(\mathbf{R}^n)$, $x \in \mathbf{R}^n$.

Let $\{T(t)\}_{t\geq 0}$ be a strongly continuous semigroup on $C_\infty(\mathbf{R}^n)$. If $\|T(t)u\| \leq \|u\|$ for all $u \in C_\infty(\mathbf{R}^n)$ and $t \geq 0$, then $\{T(t)\}_{t\geq 0}$ is a contraction semigroup. A strongly continuous positive contraction semigroup on $C_\infty(\mathbf{R}^n)$ is called a *Feller semigroup* on $\mathbf{R}^n$. We have an integral representation of the operators which form a Feller semigroup, analogous with the one of convolution semigroup (see [8], [5]).

THEOREM 2.2. *Let $\{T(t)\}_{t\geq 0}$ be a Feller semigroup on $\mathbf{R}^n$. For any $t \geq 0$ there exists a unique function $p_t : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{C}$ measurable, locally bounded and such that for any fixed $x \in \mathbf{R}^n$, $\xi \to p_t(x, \xi)$ is a continuous positive definite function with the property that for any $u \in S(\mathbf{R}^n)$,*

$$T(t)u(x) = (2\pi)^{-(n/2)} \int_{\mathbf{R}^n} \mathrm{e}^{\mathrm{i}x\cdot\xi} p_t(x, \xi)\widehat{u}(\xi)\mathrm{d}\xi.$$

For $u \in S(\mathbf{R}^n)$, the infinitesimal generator A of $\{T(t)\}_{t\geq 0}$ is

$$Au(x) = (2\pi)^{-(n/2)} \int_{\mathbf{R}^n} \mathrm{e}^{\mathrm{i}x\cdot\xi} a(x, \xi)\widehat{u}(\xi)\mathrm{d}\xi,$$

where $a : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{C}$,

$$a(x, \xi) = \frac{\mathrm{d}}{\mathrm{d}t}\, p_t(x, \xi)\Big|_{t=0}.$$

Moreover, we deduce the following result.

PROPOSITION 2.3. *Let $\{T(t)\}_{t\geq 0}$ be a Feller semigroup on $\mathbf{R}^n$. For any $t \geq 0$ and $u \in C_b^2(\mathbf{R}^n)$, $T(t)u(x) = C(t)u(x) + D(t)u(x)$, with*

$$C(t)u(x) := \sum_{i,j=1}^{n} a_{ij}^{(t)}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{n} b_i^{(t)}(x) \frac{\partial u}{\partial x_i}(x) + \gamma^{(t)}(x)u(x)$$

and

$$D(t)u(x) := \int_{\mathbf{R}^n} N^{(t)}(x, \mathrm{d}y) \left\{ u(y) - \sigma_x^{(t)}(y) \left[u(x) + \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}(x) \cdot (y_i - x_i) \right] \right\},$$

where $\gamma^{(t)}(x) = c^{(t)}(x) + d^{(t)}(x) + 1$, $a_{ij}^{(t)}$, $b_i^{(t)}$, $c^{(t)}$, $d^{(t)}$ *are continuous functions,* $a_{ij}^{(t)} = a_{ji}^{(t)}$, $\sum_{i,j=1}^{n} a_{ij}^{(t)}(x)\xi_i\xi_j \geq 0$, $c^{(t)} \leq 0$, $\sigma_x^{(t)}$ *is a certain cutt-off function and* $N^{(t)}(x, \mathrm{d}y)$ *is a certain Lévy kernel such that*

$$d^{(t)}(x) + \int_{\mathbf{R}^n} N^{(t)}(x, \mathrm{d}y)\{1 - \sigma_x^{(t)}(y)\} \leq 0,$$

for all $x \in \mathbf{R}^n$.

Proof. Indeed, $T(t) - I$ satisfies the positive maximum principle on $C_\infty(\mathbf{R}^n)$ for every $t \geq 0$. The above formula follows from the last assertion of Lemma 3.3 ([3], p. 2–34) and Corollary 3 ([3], p. 2–10). □

3. FRACTIONAL CAUCHY PROBLEM

For a function g with $\widetilde{g}(s) := \int_0^\infty \mathrm{e}^{-st} g(t)\,\mathrm{d}t$ the Laplace transform, we define *the Caputo fractional derivative in time* $\frac{\partial^\beta}{\partial t^\beta} g(t)$ as the inverse Laplace transform of $s^\beta \widetilde{g}(s) - s^{\beta-1} g(0)$. On the other hand,

$$D_t^\beta g(t) = \frac{\mathrm{d}^m}{\mathrm{d}t^m} \int_0^t \frac{(t-u)^{m-\beta-1}}{\Gamma(m-\beta)} g(u)\,\mathrm{d}u, \quad m = [\beta]$$

is *the Riemann-Liouville fractional derivative* of order β. We have

$$\frac{\partial^\beta}{\partial t^\beta} g(t) = \int_0^t \frac{(t-u)^{m-\beta-1}}{\Gamma(m-\beta)} g^{(m)}(u)\,\mathrm{d}u, \quad m = [\beta].$$

If β is a positive integer then $D_t^\beta = \frac{\partial^\beta}{\partial t^\beta}$ is the usual derivative operator.

We consider the fractional Cauchy problem

$$\frac{\partial^\beta}{\partial t^\beta} u(x,t) = Au(x,t), \quad u(x,0) = f(x),$$

where $0 < \beta < 1$, $t \geq 0$ and A is the generator of bounded continuous semigroup $\{T(t)\}_{t \geq 0}$ on the Banach space X. We observe that $p(x,t) = T(t) f(x)$ is the unique solution to the abstract Cauchy problem

$$\frac{\partial}{\partial t} p(x,t) = Ap(x,t), \quad p(x,0) = f(x),$$

for any f in the domain of A and $t > 0$.

We note that the fractional Cauchy problem can be written in several equivalent forms (see [1] and [2]).

Proposition 3.1. *Assume* $0 < \beta < 1$. *Let* A *be the generator of a strongly continuous semigroup* $\{T(t)\}_{t \geq 0}$ *on the Banach space* X *and* $g \in$

$C([0,\infty)\times X)$ be Laplace transformable. Then for all $f\in X$ the following are equivalent:

(i) *For all $t>0$, the Riemann-Liouville derivative of g exists, $g(t)\in D(A)$, the Laplace transform of $D_t^{\beta}g(t)$ exists, and*

$$D_t^{\beta}g(t)=Ag(t)+\frac{t^{-\beta}}{\Gamma(1-\beta)}f.$$

(ii) *For all $t>0$, the Caputo derivative of g exists, $g(t)\in D(A)$, the Laplace transform of $\frac{\partial^{\beta}}{\partial t^{\beta}}g(t)$ exists, and*

$$\frac{\partial^{\beta}}{\partial t^{\beta}}g(t)=Ag(t),\quad g(0)=f.$$

(iii) *For all $t>0$, the function g is differentiable, $g(t)\in D(A)$, the Laplace transform of $\frac{\partial}{\partial t}g(t)$ exists, and*

$$\frac{\partial}{\partial t}g(t)=D_t^{1-\beta}Ag(t),\quad g(0)=f.$$

(iv) *The function $g(t)$ is analytic on $0<t<\infty$, satisfies $\|g(t)\|\le M\mathrm{e}^{\omega t}$ on $0<t<\infty$ for some M, $\omega\ge 0$ and*

$$g(t)=\int_0^{\infty}\frac{t}{\beta s^{1+1/\beta}}g_{\beta}\left(\frac{t}{s^{1/\beta}}\right)T(s)f\mathrm{d}s,$$

where g_{β} is such that

$$\int_0^{\infty}\mathrm{e}^{-\lambda t}g_{\beta}(t)\,\mathrm{d}t=\mathrm{e}^{-\lambda^{\beta}}.$$

In the framework of Proposition 3.1, we define the family of bounded, strongly continuous linear operators on X,

$$S(t)h(x):=\int_0^{\infty}\frac{t}{\beta s^{1+1/\beta}}g_{\beta}\left(\frac{t}{s^{1/\beta}}\right)T(s)f(x)\,\mathrm{d}s,\quad t\ge 0.$$

On account of (iv) the function $g(t)=S(t)f$ defines a solution to the fractional Cauchy problem for any initial condition $f\in X$ and this solution depends continuously on the initial condition f.

In the sequel we consider $X=C_{\infty}(\mathbf{R}^n)$ and A the generator of a Feller semigroup $\{T(t)\}_{t\ge 0}$ on $\mathbf{R}^n$. Then by Theorem 2.2, for any $t\ge 0$ there exists a unique function $p_t:\mathbf{R}^n\times\mathbf{R}^n\to\mathbf{C}$ measurable, locally bounded and such that for any fixed $x\in\mathbf{R}^n$, $\xi\to p_t(x,\xi)$ is a continuous positive definite function with the property that for any $f\in S(\mathbf{R}^n)$,

$$T(t)f(x)=(2\pi)^{-(n/2)}\int_{\mathbf{R}^n}\mathrm{e}^{\mathrm{i}x\cdot\xi}p_t(x,\xi)\,\widehat{f}(\xi)\,\mathrm{d}\xi.$$

We suppose that the integral $\int_0^\infty \frac{1}{s^{1+1/\beta}} g_\beta\left(\frac{t}{s^{1/\beta}}\right) p_s(x,\xi)\,\mathrm{d}s$ is convergent for all x and ξ. The following statement is true.

PROPOSITION 3.2. *For any* $t \geq 0$ *and* $f \in S(\mathbf{R}^n)$,

$$S(t) f(x) = (2\pi)^{-(n/2)} \int_{\mathbf{R}^n} \mathrm{e}^{\mathrm{i}x\cdot\xi} q_t(x,\xi) \widehat{f}(\xi)\,\mathrm{d}\xi,$$

where $q_t : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{C}$ *is measurable, locally bounded and such that for any fixed* $x \in \mathbf{R}^n$, $\xi \to q_t(x,\xi)$ *is a continuous positive definite function.*

Proof. In the formula of the definition of $S(t) f(x)$ we apply the above thoughts for the semigroup $\{T(t)\}_{t\geq 0}$ on $\mathbf{R}^n$. We define

$$q_t(x,\xi) := \frac{t}{\beta} \int_0^\infty \frac{1}{s^{1+1/\beta}} g_\beta\left(\frac{t}{s^{1/\beta}}\right) p_s(x,\xi)\,\mathrm{d}s$$

and we observe that q_t satisfies the required properties. □

Remark 3.3. Using the relation from Proposition 2.3, we obtain the "structure" of each operator $S(t)$, $t \geq 0$. Since $T(t) f(x) = C(t) f(x) + D(t) f(x)$, we have

$$S(t) f(x) = \int_0^\infty \frac{t}{\beta s^{1+1/\beta}} g_\beta\left(\frac{t}{s^{1/\beta}}\right) C(s)f(x)\mathrm{d}s + \\ + \int_0^\infty \frac{t}{\beta s^{1+1/\beta}} g_\beta\left(\frac{t}{s^{1/\beta}}\right) D(s)f(x)\mathrm{d}s.$$

Thus we can interpret the solution $g(t) = S(t) f$ of the fractional Cauchy problem for the initial condition f as the sum of "diffusion part" and "Lévy part".

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