(Knowledge for Development)

KIBABII UNIVERSITY

UNIVERSITY EXAMINATIONS

2022/2023 ACADEMIC YEAR

FORTH YEAR FIRST SEMESTER

SUPPLEMENTARY EXAMINATION

FOR DEGREE OF BACHELOR OF SCIENCE MATHEMATICS

COURSE CODE: MAP 413

COURSE TITLE: FUNCTIONAL ANALYSIS

DATE: 14/11/2022 **TIME:** 8:00 AM - 10:00 AM

INSTRUCTIONS TO CANDIDATES

Answer question ONE and any other two questions

TIME: 2 Hours

QUESTION ONE (20 MARKS)

a) Define a metric space

b) Show that the sequence space $\mathcal{L}^p$ with d defined by $d(x,y) = (\sum_{j=1}^{\infty}|x_j - y_j|^p)^{\frac{1}{p}}$ is a metric space

c) Given $\{x_1, x_2, \dots, x_n\}$ are a linearly independent set of vectors in a anormed space X (of any dimension) show that there is a number $c > 0$ such that for every choice of scalars $\propto_1, \dots, \propto_n$ we have $||\propto_1 x_1 + \cdots + \propto_n x_n|| \geq c(|\propto_n| + \cdots + |\propto_n|)$

QUESTION TWO (20 MARKS)

a) Show that the three-dimensional space $\mathbb{R}^3$ with the distance

$d(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}$ is a metric space

b) Show that the following

i. Holder's inequality for sums $\sum_{j=1}^{\infty}|x_j y_j| \leq (\sum_{j=1}^{\infty}|x_j|^p)^{\frac{1}{p}} (\sum_{j=1}^{\infty}|y_j|^q)^{\frac{1}{q}}$

ii. Minkowski inequality for sums $(\sum_{j=1}^{\infty}|x_j + y_j|^p)^{\frac{1}{p}} \leq (\sum_{j=1}^{\infty}|x_j|^p)^{\frac{1}{p}} + (\sum_{j=1}^{\infty}|y_j|^p)^{\frac{1}{p}}$

QUESTION THREE (20 MARKS)

a) Show that the space $\mathcal{L}^{\infty}$ is not separable

b) Show that a subspace M of a complete metric space X is itself complete if and only if the set M is closed in X

c) Show that the space $(\mathcal{L}^{\infty}, d)$ with d defined by $d(x,y) = sup_{j\epsilon N}|x_i - y_j|$ is a metric space

QUESTION FOUR (20 MARKS)

a) Show that the Euclidean space $\mathbb{R}^n$ is complete

b) Show that the space $\mathcal{L}^p$ $1 \leq p \leq \infty$ is separable

c) Show that a metric d induced by a norm on a normed space X satisfies

i. $d(x + a, y + a) = d(x,y)$

ii. $d(ax, ay) = |a|d(x,y)$

QUESTION FIVE (20 MARKS)

a) Show that a compact subset M of a metric space is closed and bounded

b) Define a linear operator T

d) Define the following terms

  i. Normed space

  ii. Banach space