In these lecture notes we report on recent breakthroughs in the functional analytic approach to maximal regularity for parabolic evolution equations, which set off a wave of activity in the last years and allowed to establish maximal L^p-regularity for large classes of classical partial differential operators and systems.

In the first chapter (Sections 2-8) we concentrate on the singular integral approach to maximal regularity. In particular we present effective Mihlin multiplier theorems for operator-valued multiplier functions in UMD-spaces as an interesting blend of ideas from the geometry of Banach spaces and harmonic analysis with R-boundedness at its center. As a corollary of this result we obtain a characterization of maximal regularity in terms of R-boundedness. We also show how the multiplier theorems “bootstrap” to give the R-boundedness of large classes of classical operators. Then we apply the theory to systems of elliptic differential operators on R^n or with some common boundary conditions and to elliptic operators in divergence form.

In Chapter II (Sections 9-15) we construct the H^∞-calculus, give various characterizations for its boundedness, and explain its connection with the “operator-sum” method and R-boundedness. In particular, we extend McIntosh’s square function method form the Hilbert space to the Banach space setting. With this tool we prove, e.g., a theorem on the closedness of sums of operators which is general enough to yield the characterization theorem of maximal L^p-regularity. We also prove perturbation theorems that allow us to show boundedness of the H^∞-calculus for various classes of differential operators we studied before. In an appendix we provide the necessary background on fractional powers of sectorial operators.