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CONJECTURE OF BANACH SPACE OPERATOR IDEALS IN NUCLEAR SPACES

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ABSTRACT

We apply the notion of Banach space operator ideals in nuclear spaces through topological vector spaces. The motivation for this study came from attempts to generalize the structure of nuclear spaces as a result of nuclear maps from functional analysis context. The compact closed structure associated with the category of relations results to nuclear ideals. Basic properties of Banach space operator ideals in relation to the structure of nuclear spaces will be demonstrated. We therefore establish a close correspondence between Banach space operator ideals and nuclear ideals through topological vector spaces.

Mathematics Subject Classification: 46A03, 46A22, 46B50.

INTRODUCTION

We give essential concepts involving definitions and other useful notions used in the sequel. Section 1.1 introduces the concept of a Banach linear operator and some of its basic properties. In section 1.2, we present the definition and properties of a locally convex topological vector space in terms of convex balanced neighborhoods of 0 as discussed by Jeremy J. B. and Ambar N. S. (2009). Finally, in section 1.3 we illustrate the notion of a nuclear space structure which was first introduced by Grothedieck (1955) and discussed by Taylor J. L. (1995) as a class of locally convex topological vector space $(l.c.t.v.s)$.

1.1 A Banach linear operator

Let $T: X \to Y$ be a linear operator between Banach spaces. Let U_X, U_Y be the closed unit balls of X, Y, respectively. Note that closed unit balls serve simultaneously the basic model for open sets and bounded sets in Banach spaces. Some properties of the linear operator T are:

(1) T is *bounded* (i.e., TU_X is a bounded subset of X)
⇔ T is *continuous* (i.e., $T^{-1}U_X$ is a 0-neighborhood of X in the norm topology)
⇔ T is *sequentially bounded* (i.e., T sends bounded sequences to bounded sequences);
(2) T is *of finite rank* (i.e., TU_X spans a finite dimensional subspace of Y)
⇔ T is *weak-norm continuous* (i.e., $T^{-1}U_X$ is a 0-neighborhood of X in the weak topology)
⇔ T sends bounded sequences to sequences spanning finite dimensional subspaces of F;
(3) T is *compact* (i.e., TU_X is totally bounded in Y)
⇔ T is continuous in the topology of uniform convergence on norm compact subsets of X' (i.e., $T^{-1}U_Y \supseteq K^0$, the polar of a norm compact subset K of the dual space X' of X)
⇔ T is *sequentially compact* (i.e., T sends bounded sequences to sequences with norm convergent subsequences); and
(4) T is *weakly compact* (i.e., TU_X is relatively weakly compact in Y)
⇔ T is continuous in the topology of uniform convergence on weakly compact subsets of X' (i.e., $T^{-1}U_Y \supseteq K^0$, the polar of a weakly compact subset K of X')
⇔ T is *sequentially weakly compact* (i.e., T sends bounded sequences to sequences with weakly convergent subsequences).

1.2 Topological Vector Spaces

A topological vector space shall mean a real or complex vector space, equipped with a Hausdorff topology for which the operations of addition and multiplication are continuous. A general open set in such a space is the union of translates of neighborhoods of 0. A topological vector space X is said to be *locally convex* if every neighborhood of 0 contains a neighborhood of 0 which is a convex set. By *a locally convex space* we shall mean a locally convex topological vector space. Remarkable, *a locally convex space* can be understood in terms of normed linear spaces, as we explain below.

If U is a convex neighborhood of 0, inside it there is a convex neighborhood of 0 which is also *balanced,* i.e. mapped into itself under multiplications by scalars of magnitude ≤ 1, Rudin W. [1987, Theorem 1.14(b)].

For a convex balanced neighborhood of 0, the function

$$\rho_U : X \to [0,\infty) : v \mapsto \rho_U(v)$$

$= inf\{t > 0 : v \in tU\}$ (1.2.1)
is a *semi-norm* in the sense that

$$\rho_U(a+b) \le \rho_U(a)+\rho_U(b),$$
$$\rho_U(ta) = |t|\rho_U(a) \quad (1.2.2)$$

for all $a, b \in X$ and scalars t.
The function ρ_U is uniformly continuous, because
$|\rho_U(a) - \rho_U(b)| \le \rho_U(a-b)$,
which is less than any given positive ϵ if $a - b \in \epsilon U$.
If the convex balanced neighborhood U of 0 is `finite' in every direction, i.e. 0 is the only vector for which all multiples lie inside U, then ρ_U is a *norm*, i.e. ρ_U is 0 only on the zero vector. Indeed, in this case the topology arises from ρ_U in the sense that every open set is the union of translates of scalar multiples of the unit ball
$U = \{v \in X : \rho_U(v) < 1\}$.
On the other hand, if U is `infinite in some direction', i.e. it contains all multiples of some non-zero vector, then ρ_U is 0 on any vector in that direction.
A sequence $(x_n)\, n \ge 1$ in a topological vector space X is said to be Cauchy with respect to a topology τ if the differences $x_n - x_m$ are all eventually in any given $\tau-$ neighborhood of 0 for large n and m. Any convergent sequence is automatically Cauchy. A topological vector space is said to be complete if every Cauchy sequence converges.
If U and V are convex balanced neighborhoods of 0 and $U \subset V$ then the identity map $X \to X$ induces a natural surjection p_{VU} on the quotients: it is the map

$$p_{VU} : X_U \to X_V : p_U(x) \mapsto p_V(x) \quad (1.2.3)$$

for all $x \in X$. This takes a picture $p_U(x)$ of x and produces a coarser picture $p_V(x)$. Note that p_{VU} is a linear contraction mapping.
If a locally convex space X is metrizable, then by Rudin W. (1987, Theorem 1.24) its topology is induced by a translation-invariant metric d for which open balls are convex, and so every neighborhood of 0 contains a locally convex neighborhood W (an open ball, for instance) for which ρ_W is a norm (of course, the topology τ_W induced by ρ_W is, in general, smaller than τ); if V is a convex balanced neighborhood of 0, with $V \subset W$, then ρ_V is also a norm.

Theorem 1.2.1 (Jeremy J. B. and Ambar N. S., (2009). *Let X be a complete, metrizable, locally convex space. Let $\mathcal{B}$ be a set of all convex, balanced neighborhoods of 0 such that every neighborhood of 0 contains as subset some neighborhood in $\mathcal{B}$, i.e. suppose $\mathcal{B}$ is a local base at 0, consisting of convex balanced sets, for the topology of X. Suppose now that associated to each $V \epsilon \mathcal{B}$ an element $x_V \in X_V = X_{/\rho^{-1}{}_V(0)}$ such that*

$$p_{WV}(x_V) = x_W \quad (1.2.4)$$

for every $W \epsilon \mathcal{B}$ for which $W \supset V$. Then there exists a unique $x \epsilon X$ such that
$p_W(X) = x_W$ for every $W \epsilon \mathcal{B}$. Moreover, for such W, the quotient X_V is a Banach space.

1.3 The nuclear space structure

Consider a real complex topological vector space H equipped with the following structure:- There is a sequence of inner-products $\langle .,. \rangle_P$, for $p \in \{0,1,2,3,\dots\}$, on $\mathcal{H}$ such that

$$\|.\|_0 \le \|.\|_1 \le \cdots \quad (1.3.1)$$

Denote H_0, the completion of $\mathcal{H}$ in the norm $\|.\|_0$ and inside this Hilbert space the completion of $\mathcal{H}$ with respect to $\|.\|_P$ be a dense subspace denoted H_P. We assume that H_0 is separable, and that $\mathcal{H}$ is the intersection of all the spaces H_P. Thus,

$$\mathcal{H} = \bigcap_{p=0}^{\infty} H_P \subset \cdots \subset H_2 \subset H_1 \subset H_0. \quad (1.3.2)$$

Furthermore, we assume that each inclusion $H_{P+1} \to H_P$ is a Hilbert Schmidt operator, i.e. there is an orthonormal basis $v_1, v_2, v_3, \dots$ in H_{P+1} for which

$$\textstyle\sum_{n=1}^{\infty} \| v_n \|^2{}_p < \infty. \quad (1.3.3)$$

The topology on $\mathcal{H}$ is the *projective limit topology* from the inclusions $\mathcal{H} \to H_P$, i.e. it is induced by the norms $\|.\|_P$. Thus an open set in this topology is the union of $\|.\|_P$-balls with p running over $\{0,1,2,3,\dots\}$. All these assumptions make $\mathcal{H}$ a *nuclear space* (Jeremy J. Becnel and Ambar N. Sengupta, 2009).
In a subtle way, we can define a nuclear space as a locally convex space such that for any seminorm p we can find a larger seminorm $p+1$ so that the natural map from $H_{P+1} \to H_P$ is *nuclear.* Informally, this means that whenever we are given the unit ball of some seminorm, we can find a "much smaller" unit ball of another seminorm inside it, or that any neighborhood of 0 contains a "much smaller" neighborhood.

Remark 1.3.1 (Taylor J. L., 1995)
An infinite-dimensional topological vector space is never locally compact, but a nuclear space is an excellent substitute in the infinite-dimensional case: the definition ensures, in particular, that an open ball in H_{P+1} has compact closure in H_P.
Note that if τ_P is the topology on $\mathcal{H}$ given by $\|.\|_P$, then by (1.3.1), the identity map
$(\mathcal{H}, \tau_{P+1}) \to (\mathcal{H}, \tau_P)$
is continuous, for $p \in \{0,1,2,3,\dots\}$ and so

$$\tau_0 \subset \tau_1 \subset \cdots \quad (1.3.4)$$

The inclusions here are strict if $\mathcal{H}$ is infinite-dimensional, because of the Hilbert-Schmidt assumption made above.
Now following the concept of Nuclear spaces introduced by Grothedieck A., (1955) and discussed by Taylor J. L., (1995) as a class of locally convex topological vector space $(l.c.t.v.s)$. Let X and Y be $l.c.s.'s$. The space $X^* \otimes Y$ is linearly isomorphic to a subspace of $L(X, Y)$, the space of continuous linear maps from X *to* Y - as follows:
if $u = \sum f_i \otimes y_i \in Y \otimes X^*$,
we assign to u the linear map $\ell_u \in L(X, Y)$ defined by
$\ell_u(x) = \sum f_i(x) y_i$
If the y_i are chosen to be linearly independent (as they may), then the only way this sum can be zero is if $f_i(x) = 0$ for every i. If this happens for every x, then the f_i are all zero and so is u. Thus, the map $u \to \ell_u$ is injective as well as (obviously) linear. Thus, we may identify $X^* \otimes Y$ with a linear subspace of $L(X, Y)$. In fact, it is obviously the linear subspace consisting of finite rank continuous linear maps from X to Y.
Now suppose that X and Y are Banach spaces. Then X^* is also a Banach space, as is the completed projective tensor product $X^* \widehat{\otimes} Y$. The map
$u \to \ell_u : X^* \otimes Y \to L(X, Y)$
is norm decreasing if $L(X, Y)$ is given the operator norm and $X^* \otimes Y$ the tensor product norm. In fact,

$$||\ell u|| = sup\{||\ell u(x)|| : ||x|| \le 1\} \le \sum ||f_i|| \, ||y_i|| ||x|| \text{ if } u = \sum f_i \otimes y_i$$

Thus

$$||\ell u|| \le inf\{\sum ||f_i|| \, ||y_i|| : u = \sum f_i \otimes y_i\} = ||u||$$

It follows that $u \to \ell u$ extends to a norm decreasing linear map $X^* \widehat{\otimes} Y \to L(X, Y)$.

Definition 1.3.2. If X and Y are Banach spaces then *a nuclear map* from $X\ to\ Y$ is an element of $L(X, Y)$ of the form ℓu for some $u \in X^* \widehat{\otimes} Y$. If X and Y are arbitrary $l.c.s.'s$ then a nuclear map from $X\ to\ Y$ is a map $\varphi \in L(X, Y)$ which factors as $\mu \circ \psi \circ \nu$ where $E\ and\ F$ are Banach spaces, $\nu \in L(X, E), \mu \in L(F, Y)$, and $\psi : E \to F$ is nuclear.

Proposition 1.3.3 (Taylor J. L., 1995). If X and Y are $l.c.s.'s$ then a linear map $\varphi \in L(X, Y)$ is nuclear if and only if it has the form

$\varphi(x) = \sum_{n=1}^{\infty} \lambda_n f_n(x) y_n$

where f_n is a sequence in X^* which converges uniformly to 0 on some 0-neighborhood $V \subset X, \{y_n\}$ is a sequence which converges to 0 in the space Y_B for some balanced, convex, bounded subset $B \subset Y$ for which Y_B is complete and $\sum |\lambda_n| < \infty$.

Definition 1.3.4. If $X\ and\ Y$ are $l.c.s.'s$ and $\varphi \in L(X, Y)$, then φ is called compact if there exist a 0-neighborhood $U\ in\ X$ such that $\varphi(U)$ has compact closure in Y.

Proposition 1.3.5 (Taylor J. L., 1995). Every nuclear map between $l.c.s.'s$ is compact.

Note that every finite rank operator is nuclear and, hence, compact.

The following proposition is obvious from the definition of nuclear map.

Proposition 1.3.6 (Taylor J. L., 1995). The composition of a nuclear map with a continuous linear map (on either side) is nuclear.

Definition 1.3.7. A nuclear space is an $l.c.s. X$ with a basis υ of convex, balanced 0-nieghborhoods such that the map $\varphi_U : X \to X_U$ is a nuclear map for each $U \in \upsilon$.

Proposition 1.3.8. (Taylor J. L., 1995). For an $l.c.s.\ X$ the following statements are equivalent:

(i) X is a nuclear space;
(ii) for every convex, balanced 0-neighborhood U there is a convex, balanced 0-neighborhood $V \subset U$ such that the map $\varphi_{UV} : X_V \to X_U$ is nuclear;
(iii) every continuous linear map from X to a Banach space is nuclear.
(iv) the map $\varphi_U : X \to X_U$ is nuclear for every convex, balanced 0-neighborhood U.

CORRESPONDENCE BETWEEN BANACH SPACE OPERATOR IDEALS AND NUCLEAR IDEALS THROUGH TOPOLOGICAL VECTOR SPACES.

We recall that a closed subspace F of a Banach space E is called an *ideal* in E if $F^{\perp}$, the annihilator of F in F^*, is the kernel of a norm one projection P on E^*. In this case P is called the *ideal projection.* The notion of an ideal in a Banach space was introduced by Godefroy, et al. (1993).

Chong Man Cho and Eun Joo Lee, (2004), illustrated that an ideal is closely linked with a Hahn-Banach extension operator in the sense that for a closed subspace F of a Banach space E a linear operator □$: F^* \to E^*$ is called a *Hahn-Banach extension operator* if □(e^*) is a norm preserving extension of e^* for all $e^* \in E^*$. It is well known that there exists a Hahn-Banach extension operator □$: F^* \to E^*$ if and only if F is an ideal in E. In this case, the Hahn-Banach extension operator □ and the corresponding ideal projection $P: E^* \to E^*$ are related by $Px^* =$ □$(x^* \backslash_F)$, where $x^* \backslash_F$ is the restriction of x^* to F.

In this section through illustrated properties of Banach operator ideals as discussed by Lima, A and Oja, E., (2004), we show that there is a close correspondence between the notions of Banach operator ideals and the structure of nuclear spaces in the sense of topological vector spaces.

Let $, X, Y, Z\ and\ W$ be Banach spaces. We denote by $L(X, Y)$ the Banach space of bounded linear operators from $X\ to\ Y$, and by $F(X, Y)$, $\overline{F(X, Y)}$, $K(X, Y)$ its subspaces of finite rank operators, approximable operators (i.e. norm limits of finite rank operators), and compact operators respectively. We illustrate these subspaces in terms of normed linear spaces as follows. By property (1) of section 1.1, $\forall\, T \in L(X, Y)$ implies that T is continuous and sequentially bounded. Property (2) of section 1.1 ensures that $\forall\, T \in F(X, Y)$ is a weak-norm continuous and sends bounded sequences to sequences spanning finite dimensional subspaces of $F(X, Y)$. Clearly, by property 1.3.5, every $T \in F(X, Y)$ is nuclear and hence compact. Supposing the subspace $F(X, Y)$ is of finite rank operators, its closure shall imply compactness property. Also since $all\ T \in F(X, Y)$ are weak norm continuous operators, properties (3) and (4) of section 1.1 therefore sets the space $\overline{F(X, Y)}$ to be weakly compact, in particular it is sequentially weakly compact. Finally, $K(X, Y)$ being a subspace of compact operators implies that it has a finite dimension and thus is a complete space. Letting this space be a balanced convex neighborhood of 0, we then can define a uniformly continuous norm on it. In the case X is metrizable, it follows by theorem 1.2.1 that there exists a unique $x \in X$ such that $\rho_W(x) = x_W$ for every W a subset of all convex, balanced neighborhood of 0. By definition 1.3.7, the space X is a nuclear space with a basis W of convex, balanced 0- neighborhoods such that the map $\psi_W: X \to X_W$ is a nuclear map for all W.

A First basic result, Proposition 2.1, is the property of an ideal illustrated as per the concepts of Hahn-Banach extension operator and tensor product spaces in relation to the nuclear space structure through topological vector spaces. In proposition 2.3 we obtain an extended version of the results in Proposition 2.1 about the locally convex space X being an ideal in a Banach space Y whenever $K(Z, Y)$ is an ideal in $K(Z, Y)$ for some convex, balanced 0-neighborhood space Z. In Proposition 2.4 we show that the converse is true whenever $K(Z, X)$ is an ideal in $K(Z, X^{**})$. Finally, in Theorem 2.5 we conclude that the question about being an ideal in separable Banach spaces is analogous to nuclear space ideals in the topological vector space sense.

Proposition 2.1. *Let X be an ideal in Y and let Z be an ideal in W. Then $X \otimes Z$ is an ideal in $Y \otimes W$.*

Proof. We shall suppose that the spaces $X, Y, Z\ and\ W$ are locally convex. For X an ideal in Y and Z an ideal in W, implies that there exists 0-neighbourhoods U and V in X and Z respectively such that the mappings $\psi(U) and\ \phi(V)$ have compact closures in Y and W. In this case both $\psi(U): X \to Y$ and $\phi(V): Z \to W$ are nuclear maps equivalent to the mappings $\psi: X^* \to Y^* and\ \phi: Z^* \to W^*$ be Hahn-Banach extension operators.

Considering the completed projective tensor products $(X \widehat{\otimes} Z)^* = I(X, Z^*)$ and $(Y \widehat{\otimes} W)^* = I(Y, W^*)$, these are linearly isomorphic to subspaces $L(X, Z)$ and $L(Y, W)$ with the maps

$u \to \ell u: (X \widehat{\otimes} Z)^* \to L(X, Z)$ and

$v \to \ell v: (Y \widehat{\otimes} W)^* \to L(Y, W)$

being norm one decreasing.

Since $\psi^* x = x,\ x \in X$, and □$^* z = z,\ z \in Z$, the map $\Phi: I(X, Z^*) \to I(Y, W^*)$ which can be expressed as a composition of a nuclear map with a continuous linear map T defined by $\Phi(T) = \psi \circ Q \circ T^{**} \circ ⍰^{*}\backslash Y$ is nuclear by proposition 1.3.6 and is clearly a Hahn-Banach extension operator thus $X \otimes Z$ *is an ideal in* $Y \otimes W$.

Corollary 2.2. *Let X be a locally convex space and Y a Banach space. The following statements are equivalent.*

(i) *X is an ideal in Y.*

(ii) $\overline{F(Z,X)}$ *is an ideal in* $\overline{F(Z,Y)}$ *for some convex balanced -0 neighborhood Z.*

In particular, $\overline{F(Z,X)}$ *is an ideal in* $\overline{F(Z,X^{**})}$ *for convex balanced -0 neighborhoods X and Z.*

Proof. $(i) \Rightarrow (ii)$ is immediate from Proposition 2.1 because the closed compact sets of $F(Z,X)$ and $F(Z,Y)$ can be canonically identified with $\underline{Z^{*} \widehat{\otimes} X}$ and $\underline{Z^{*} \widehat{\otimes} X}$.

$(ii) \Rightarrow (i)$. Suppose $\overline{F(Z,X)}$ is an ideal in $\overline{F(Z,Y)}$. Let F be a finite dimensional subspace of Y. Then F is injective and is obviously the linear subspace consisting of finite rank continuous linear maps from X to Y. Consequently, we can define a linear operator $⍰: Z^{*} \widehat{\otimes} X \rightarrow Z^{*} \widehat{\otimes} Y$ to be a Hahn Banach extension operator by letting $⍰(z^{*})$ be a norm one preserving extension of Z^{*}, $\forall\, z^{*} \in Z^{*}$. The set $\overline{F(Z,Y)}$ is a locally convex space with the subspace $Z^{*} \otimes Y$ a basis of a balanced convex bounded 0-neighbourhood $\forall\, y \in F(Z,Y)$ so that $V : Z^{*} \widehat{\otimes} Y \rightarrow \overline{F(Z,Y)}$ is also norm 1 projection

Now define a map $U : F(Z,X) \rightarrow \overline{F(Z,X^{**})}$ by $Uy : F(Z,X) \rightarrow F(Z,Y)$. By proposition 1.3.8, the map Uy is nuclear and thus U ``locally 1-complements'' X in Y.

However, by the proof of the implication $(ii) \Rightarrow (i)$, we also have the similar result in the case of compact operators.

Proposition 2.3. *Let X be a locally convex space and Y a Banach space* Y. *If* $K(Z,X)$ *is an ideal in* $K(Z,Y)$ *for some convex balanced 0-neighborhood* Z, *then X is an ideal in Y.*

Proposition 2.4. *Let X a locally convex space and Y a Banach space and assume that* $K(Z,X)$ *is an ideal in* $K(Z,X^{**})$ *for some convex balanced 0- neighborhood* Z. *Then X is an ideal in Y if and only if* $K(Z,X)$ *is an ideal in* $K(Z,Y)$.

Proof. In view of Proposition 2.3 we only need to prove the "only if" part.

Let ⍰ be a continuous linear map from X to Y. Since X is a locally convex space and Y a Banach space, then ⍰ is nuclear. Also let $K(Z,X)^{*}$ and $K(Z,X^{**})$ be balanced *-0 neighborhoods* such that there is a nuclear map $\Phi: K(Z,X)^{*} \rightarrow K(Z,X^{**})^{*}$. This nuclear map is equivalent to the Hahn-Banach extension operator for finite dimensional convex balanced 0- neighborhood spaces. Let Ψ be a composition of the nuclear map Φ with the continuous linear map ⍰, defined by

$(\Psi f)(T) = (\Phi f)(\phi^{*}\backslash Y \circ T)$,

$f \in K(Z,X^{*})^{*},\ T \in K(Z,Y)$,

then Ψ is nuclear and hence compact by proposition 1.3.8.

The composite nuclear map Ψ consequently is a translation-invariant metric which induces a topology on $K(Z,X)^{*}$ for the convex balanced 0- neighborhood space $K(Z,Y)$ with $\|\Psi\| \leq 1$. Hence Ψ is a Hahn Banach extension operator.

Theorem 2.5. *Let* X,Z *be locally convex spaces and Y a Banach space such that X is a basis of convex* balanced 0-neighborhood subset of *Z. Then* $K(Y,X)$ *is an ideal in* $K(Y,Z)$ *if and only if* $K(W,X)$ *is an ideal in* $K(W,Z)$ *for every separable ideal* W *in* Y.

Proof. Assume that the separable ideal W has a nuclear structure and for some sequence of inner products $<.,>_{P}$ for $p \in (0,1,2,3, \ldots\ldots)$ on $K(Y,Z)$ such that $\|.\|_0 \leq \|.\|_1 \leq \cdots$ and $K(Y,X)$ a subspace in the completion of $K(Y,Z)$. Assume that $K(Y,X)$ is separable in the completion space $K(Y,Z)$. With reference to the definition of the nuclear structure in section 1.3, denote the space $K(Y,Z)$ by $\mathcal{H}$ which is the intersection of all the compact spaces and $K(Y,X)$ by H_P, a dense subspace in $\mathcal{H}$.

Let $:\Phi: K(Y,X)^{*} \rightarrow K(Y,Z)^{*}$ and $⍰: W^{*} \rightarrow Y^{*}$ be Hahn-Banach extension operators equivalent to Hilbert Schimidt operators for some orthonormal basis in $K(Z,X)^{*}$ and W^{*}. Let $f \in K(W,X)^{*}$ be any finite dimensional nuclear subspace and and $S \in K(Z,X)$ a compact space, then the composite space $\bar{f}(S) = f(S\backslash W)$ is a projective limit topology from inclusions $K(W,Y) \rightarrow K(W,Z)$ for a separable ideal W. Thus we can define a composite nuclear map Ψ from $K(W,X)^{*} \rightarrow K(W,Z)^{*}$ by

$(\Psi f)(T) = \left(\Phi \bar{f}\right)(T^{**} \circ \phi^{*}\backslash Y),\ f \in K(W,Z)^{*},\ T \in K(W,Z).$

With Ψ being a decreasing norm 1 projection on W^{*}. Now for $T \in K(W,X)$, we have $T^{**} \circ \phi^{*}\backslash Y \in K(Y,X)$ and therefore

$(\Psi f)(T) = \left(\Phi \bar{f}\right)(T^{**} \circ \phi^{*}\backslash Y) = \bar{f}(T^{**} \circ \phi^{*}\backslash Y)$

$= f(T^{**} \circ \phi^{*}\backslash W) = f(T^{**}\backslash W) = f(T).$

Hence Ψ is a Hahn-Banach extension operator equivalent to the continuous identity map in inclusion closed compact spaces $K(W,X)$ of $K(W,Z)$.

Conversely, assume that $K(W,X)$ is an ideal in $K(W,Z)$ for every separable ideal W with a nuclear structure in Z. Let $F \subseteq K(Y,Z)$ be a finite dimensional subspace. By remark 1.3.1 the set $\{T^{*}z^{*}: T \in F,\ z^{*} \in Z^{*}\}$ is locally compact and separable- by the Hilbert Schimidt assumption. Also by a theorem due to Sims and Yost (1989), we can find a separable ideal W *in* Z with a Hahn-Banach extension operator $⍰: W^{*} \rightarrow Y^{*}$ such that $\{T^{*}z^{*}: T \in F,\ z^{*} \in Z^{*}\} \subseteq \phi(W^{*})$. If we let I be an identity map from the separable locally compact nuclear structure W to the Banach space Y, then $I^{*}: Y^{*} \rightarrow W^{*}$ is the restriction operator and we get

$I_{⍰(W^{*})} = (⍰ \circ I^{*})\backslash_{⍰(W^{*})}$. On W we can find a norm 1 decreasing linear operator $V: F_W \rightarrow K(W,X)$ such that $V(S) = S$ for every subspace S of all intersection of spaces of finite rank continuous linear maps on W. Similarly, we can define another finite rank continuous norm 1 decreasing linear operator U as a composite linear map on W by $U(T) = (V(T \circ I))^{**} \circ \phi^{*}\backslash Z$. Then for any $y \in Y,\ z^{*} \in Z^{*}$, and T, a finite rank continuous linear operator such that $T^{*}z^{*} \in \phi(W^{*})$, we get

$z^{*}(U(T)y) = z^{*}((V(T \circ I))^{**} \circ (\phi^{*}y))$

$= z^{*}((T^{**} \circ I^{**})\,(\phi^{*}y))$

$= z^{*}((T^{**} \circ I^{**} \circ \phi^{*})y)$

$= ((\phi \circ I^{*} \circ T^{*})\,(z^{*}))(y)$

$= (T^{*}z^{*})(y) = y^{*}(Tz).$

Thus U is linearly isomorphic. Hence $K(Z,X)$ is an ideal in $K(Z,Y)$.

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