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Mixed finite difference and Galerkin methods for solving Burgers equations using interpolating scaling functions

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The current paper proposes a technique for the numerical solution of Burgers equations. The method is based on finite difference formula combined with the Galerkin method, which uses the interpolating scaling functions. Several test problems are given, and the numerical results are reported to show the accuracy and efficiency of the new algorithm.

Keywords: Burgers equation; KdV–Burgers and coupled Burgers equations; interpolating scaling functions; mixed finite difference method; operational matrix of derivative

1. Introduction

Nonlinear partial differential equations arise in a large number of mathematical and engineering problems. Systems of nonlinear partial differential equations have attracted much attention in studying solid state physics, fluid mechanics, chemical, propagation of undular bores in shallow water waves [1], propagation of waves in elastic tube filled with a viscous fluid [2], and plasma physics [3]. Burgers equation is one of the well-known equations in mathematics and physics. This equation has been found to describe various kinds of phenomena such as the mathematical model of turbulence [4] and the approximate theory of flow through a shock wave traveling in a viscous fluid [5]. The Korteweg–de Vries–Burgers (KdV–Burgers) equation is a 1-D generalization of the model description of the density and velocity fields that takes into account pressure forces as well as the viscosity and the dispersion. Several numerical methods are used such as Chebyshev spectral collocation method [6], meshfree interpolation method [7], modified extended backward differentiation formula [8], direct variational methods [9], and so on to solve these equations [10, 11].

In this paper, mixed finite difference [12] and Galerkin methods are used to solve the 1-D, KdV [13], and coupled Burgers equations with interpolating scaling functions (ISFs). Burgers equation in this paper is represented in three types as

($E1$) 1-D Burgers equation

$$u_t + \alpha u u_x - \upsilon u_{xx} = 0, \quad (x,t) \in [a,b] \times [0,T], \tag{1.1}$$

with the initial and boundary conditions

$$u(x,0) = f(x), \quad x \in [a,b], \tag{1.2}$$

$$u(x,t) = g(t), \quad (x,t) \in [a,b] \times [0,T], \tag{1.3}$$

respectively, where α and υ are arbitrary constants.

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($E2$) KdV–Burgers equation

$$u_t + \alpha u u_x - \upsilon u_{xx} + \mu u_{xxx} = 0, \quad (x,t) \in [a,b] \times [0,T], \tag{1.4}$$

with the initial condition

$$u(x,0) = \tilde{f}(x), \quad x \in [a,b], \tag{1.5}$$

and the boundary conditions

$$u(x,t) = \tilde{g}(t), \quad u_x(x,t) = \tilde{h}(t), \quad (x,t) \in [a,b] \times [0,T], \tag{1.6}$$

where α, υ, and μ are arbitrary constants.
($E3$) Coupled Burgers equations

$$\begin{cases} u_t - u_{xx} + 2uu_x + \alpha(uv)_x = 0, \\ v_t - v_{xx} + 2vv_x + \beta(uv)_x = 0, \end{cases} \tag{1.7}$$

with the boundary and initial conditions

$$\begin{cases} u(a,t) = f_1(t), \quad v(a,t) = g_1(t), \quad t > 0, \\ u(b,t) = f_2(t), \quad v(b,t) = g_2(t), \quad t > 0, \end{cases} \tag{1.8}$$

$$\{ \; u(x,0) = f(x), \quad v(x,0) = g(x), \quad x \in [a,b], \tag{1.9}$$

respectively, where α and β are arbitrary constants.

$E1$ is the simplest nonlinear equation for diffusive waves in fluid dynamics. Burgers equation arises in many physical problems including 1-D turbulence, sound waves in a viscous medium, shock waves in a viscous medium [14, 15], waves in fluid filled viscous elastic tubes, and magnetohydrodynamic (MHD) waves [16] in a medium with finite electrical conductivity [17]. The Burgers equation is similar to the 1-D Navier–Stokes equation without the stress term [17]. It is also used in the description of the variation in vehicle density in highway traffic [14, 18]. It is one of the fundamental equations in fluid mechanics. The Burgers equation demonstrates the coupling between diffusion u_{xx} and the convection process uu_x. Burgers introduced this equation to capture some of the features of turbulent fluid in a channel caused by the interaction of the opposite effects of convection and diffusion [19]. It is also used to describe the structure of shock waves, traffic flow, and acoustic transmission [20]. A great deal of effort has been expended in the last few years to compute efficiently the numerical solution of the Burgers equation for small and large values of the kinematic viscosity. So far, various numerical algorithms such as automatic differentiation method [21], Galerkin finite element method [22], cubic B-splines collocation method [23, 24], spectral collocation method [6, 25], sinc differential quadrature method [26], polynomial-based differential quadrature method [27], quartic B-splines differential quadrature method [28], and quartic B-splines collocation method [29] are proposed. Veksler and Zarmi [14] constructed fronts from exponential wave solutions of the Lax pair associated with the Burgers equation. In [14], a useful study was introduced to handle the perturbed and the unperturbed Burgers equations. The normal form analysis of the perturbed equation and a number of aspects of the freedom were investigated in [14].

The KdV–Burgers equation is a 1-D generalization of the model description of the density and velocity fields that takes into account pressure forces as well as the viscosity and the dispersion. It may be a more flexible tool for physicists than the Burgers equation. Equation (1.4) has been derived as a model for the propagation of weakly nonlinear dispersive long waves in some physical contexts when dissipative effects occur [30, 31]. The global well-posedness of (1.4) and the generalized KdV–Burgers equation have been studied by many authors (see [32, 33] and the reference therein). In [32] Molinet and Ribaud studied (1.4) and showed that (1.4) is globally well-posed in $H^s(s > -1)$.

Several numerical methods to solve this equation have been given such as algorithms based on Adomian decomposition method [34, 35], finite difference method [34]. Also, Galerkin quadratic B-spline finite element method [36] and spectral collocation method [37] have been used to obtain numerical solutions of some nonlinear evolution equations [38].

The coupled Burgers system was derived by Esipov [39] to study the model of polydispersive sedimentation. System (1.7) arises in various physical contexts, for example, it is a simple model of sedimentation or evolution of scaled volume concentrations of two kinds of particles in fluid suspensions or colloids, under the effect of gravity [40]. Also, it is a simple model coming from the theory of 1-D MHD turbulence. Because the full MHD equations [16] are too complicated to investigate the small scale structure of the MHD turbulence, it is necessary to present simple models, which contain essential features of the MHD turbulence [41]. The system (1.7) is the simplest possible set of equations, which allow 'Alfvenization', that is, the interchange of magnetic and fluid energies. It can be derived from the full MHD equations [42] when the plasma density length scales are much longer than those of the magnetic field, resulting, to leading order, in a constant density, see [43] for details. Equation (1.7) may also model the opposite limit of a fluid–dominated (i.e., unmagnetized) system [44]. For broader applicability of (1.7), we refer the interested reader to references [45–47] and so on.

In recent years, several studies for the coupled linear and nonlinear initial/boundary value problems have been appeared in the literature. Numerical algorithms such as harmonic differential quadrature finite differences coupled approach [48] and conjugate filter approach [49] are available for obtaining approximate solutions of coupled equations as well as nonlinear differential equations.

Also, an application of meshfree interpolation method [50, 51] for the numerical solution of the coupled nonlinear partial differential equations is proposed in [7]. Authors of [6] have obtained the approximate solution of the viscous coupled Burgers equations using cubic-spline collocation method. The equation has been solved by Dehghan *et al*. [52] using a Pade technique, and authors of [53] have used Fourier pseudospectral method [54] to find numerical solution of the equation. Variational iteration method [55, 56] has been presented for solving the coupled viscous Burgers equations by Adbou and Soliman [57]. Also, the differential transformation method [58], Bäcklund transformation and similarity reduction technique [59] and cubic B-spline collocation scheme on the uniform mesh [60] are proposed to solve these equations.

In this work, the interpolating scaling functions (ISFs) are used to solve Burgers equation. This type of function is obtained by multiresolution analysis and is called father wavelets. Solving Burgers equation is the aim of this paper by mixed finite difference and Galerkin method (MFDGM). For this purpose, we apply finite difference method [61] for variable t and then use the ISFs to solve the ordinary differential equation obtained from the finite difference method.

The outline of this paper is as follows. In Section 2, we describe the interpolation scaling functions and their properties and construct their operational matrix of derivatives. In Section 3, the proposed method is used to approximate the solution of the problem. Also, the stability of this method is described in Subsection 3.4. In Section 4, the numerical results of applying the method of this article on some test problems for the 1-D, KdV, and coupled Burgers equations are presented. Finally, a conclusion is drawn in Section 5.

2. The interpolating scaling functions

Suppose that $L_k(t)$, $k = 0, \ldots, r-1$, are the Lagrange interpolating polynomials given as [62, 63]

$$L_k(t) = \prod_{i=0, i\neq k}^{r-1} \left(\frac{t-\tau_i}{\tau_k - \tau_i} \right),$$

and ω_k, $k = 0, \ldots, r-1$, are the Gauss–Legendre quadrature weights given as

$$\omega_k = \frac{2}{rP'_r(\tau_k)P_{r-1}(\tau_k)},$$

where for any fixed nonnegative integer number r, P_r is the Legendre polynomial of order r and for $k = 0, \ldots, r-1$, τ_k are the roots of P_r. Now, ISFs are given by [64, 65]

$$\phi^k(t) = \begin{cases} \sqrt{\frac{2}{\omega_k}} L_k(2t-1), & t \in [0,1], \\ 0, & \text{otherwise}. \end{cases}$$

2.1. *The function approximation*

For any two fixed nonnegative integer numbers r and n, a function $f(t) \in L^2[a,b]$ may be represented by ISF expansion as

$$f(t) \approx \sum_{m=a}^{b-1} \sum_{k=0}^{r-1} \sum_{l=0}^{2^n-1} s_{nl}^{km} \phi_{nl}^{km}(t) = S^T \Phi(t), \tag{2.1}$$

where

$$\begin{aligned} S &= \left[s_{n0}^{0,a}, \ldots, s_{n0}^{r-1,a} | \ldots | s_{n,2^n-1}^{0,a}, \ldots, s_{n,2^n-1}^{r-1,a} | \ldots\ldots | s_{n0}^{0,b-1}, \ldots, s_{n0}^{r-1,b-1} | \ldots | s_{n,2^n-1}^{0,b-1}, \ldots, s_{n,2^n-1}^{r-1,b-1} \right]^T, \\ \Phi(t) &= \left[\phi_{n0}^{0,a}(t), \ldots, \phi_{n0}^{r-1,a}(t) | \ldots | \phi_{n,2^n-1}^{0,a}(t), \ldots, \phi_{n,2^n-1}^{r-1,a}(t) | \ldots \right. \\ &\quad \left. \ldots | \phi_{n0}^{0,b-1}(t), \ldots, \phi_{n0}^{r-1,b-1}(t) | \ldots | \phi_{n,2^n-1}^{0,b-1}(t), \ldots, \phi_{n,2^n-1}^{r-1,b-1}(t) \right]^T, \end{aligned} \tag{2.2}$$

and the coefficients s_{nl}^{km} are computed as

$$s_{nl}^{km} = \int_0^1 f(t) \phi_{nl}^{km}(t) dt = \int_{h_l^m}^{h_{l+1}^m} f(t) \phi_{nl}^{km}(t) dt,$$

where

$$h_l^m = \frac{l + m2^n}{2^n}, \qquad l = 0, \ldots, 2^n - 1, \; m = a, \ldots, b-1.$$

These coefficients may be computed using Gauss–Legendre quadrature [65] as

$$\begin{aligned} s_{nl}^{km} &\approx 2^{-n/2} \sqrt{\frac{\omega_k}{2}} f\left(2^{-n}\left(\hat{\tau}_k + l + m2^n\right)\right), \\ k &= 0, \ldots, r-1, \; l = 0, \ldots, 2^n - 1, \; m = a, \ldots, b-1, \end{aligned} \tag{2.3}$$

where

$$\hat{\tau}_k = \frac{\tau_k + 1}{2}.$$

Let

$$\begin{aligned}\phi_{nl}^{k,m}(t) &= 2^{(n/2)}\phi^k\left(2^n t - l - m2^n\right), \\ &k = 0, \ldots, r-1,\ l = 0, \ldots, 2^n - 1,\ m = a, \ldots, b-1,\end{aligned} \tag{2.4}$$

where n is a fixed nonnegative integer number, then we have the following orthonormality conditions [65]

$$\int_0^1 \phi_{nl}^{k,m}(t)\phi_{n\acute{l}}^{\acute{k}\acute{m}}(t)dt = \delta_{l\acute{l}}\delta_{k\acute{k}}\delta_{m\acute{m}},$$

$$k, \acute{k} = 0, \ldots, r-1, \quad l, \acute{l} = 0, \ldots, 2^n - 1,\ m, \acute{m} = a, \cdots b-1.$$

2.2. The operational matrix of the derivative

Suppose that the derivative of $f(t)$ in (2.1) be given by

$$\frac{d}{dt}f(t) \approx \sum_{m=a}^{b-1}\sum_{k=0}^{r-1}\sum_{l=0}^{2^n-1} \tilde{s}_{nl}^{km}\phi_{nl}^{km}(t) = \tilde{S}^T\Phi(t), \tag{2.5}$$

where $\tilde{S}$ is a vector defined similarly to (2.2). We express relation between S and $\tilde{S}$ by

$$\tilde{S} = DS, \tag{2.6}$$

where D is the operational matrix of the derivative for the ISFs and can be shown as a block tridiagonal matrix as [64]

$$D = 2^n \begin{bmatrix} \underline{R} & H & & & & \\ -H^T & R & H & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & -H^T & R & H \\ & & & & -H^T & \overline{R} \end{bmatrix}_{N\times N},$$

where, each block is an $r \times r$ matrix and $N = (b-a)r2^n$. Also for $k, i = 1, \ldots, r$, we have

$$\begin{aligned}[\underline{R}]_{ki} &= \frac{1}{2}\phi^i(1)\phi^k(1) - \phi^i(0)\phi^k(0) - q_{ki}, \\ [R]_{ki} &= \frac{1}{2}\phi^i(1)\phi^k(1) - \frac{1}{2}\phi^i(0)\phi^k(0) - q_{ki}, \\ [\overline{R}]_{ki} &= \phi^i(1)\phi^k(1) - \frac{1}{2}\phi^i(0)\phi^k(0) - q_{ki}, \\ [H]_{ki} &= \frac{1}{2}\phi^i(0)\phi^k(1).\end{aligned}$$

The operational matrix of the derivative is exact for polynomials up to degree $r-1$.

3. The mixed finite difference and Galerkin method

In this section, we solve nonlinear partial differential equations $E1$, $E2$, and $E3$ on a bounded domain. For this purpose, we use finite difference method for one variable to reduce these equations to a system of ordinary differential equations [66], then we solve this system and find the solution of the given partial differential equation at the points $t_i, i = 0, \ldots, \frac{1}{\delta t}$.

3.1. The mixed finite difference method for equation E1

Let us first consider the 1-D Burgers equation that has the form

$$u_t + \alpha u u_x - \upsilon u_{xx} = 0, \quad (x,t) \in [a,b] \times [0,T], \tag{3.1}$$

with the initial condition

$$u(x,0) = f(x), \quad x \in [a,b], \tag{3.2}$$

and the boundary conditions

$$u(a,t) = g_1(t), \; u(b,t) = g_2(t), \quad t \in [0,T]. \tag{3.3}$$

We discretize (3.1) according to the following θ-weighted scheme

$$\frac{u^{h+1} - u^h}{\delta t} + \theta\left(\alpha u^{h+1} u_x^{h+1} - \upsilon u_{xx}^{h+1}\right) + (1-\theta)\left(\alpha u^h u_x^h - \upsilon u_{xx}^h\right) = 0, \tag{3.4}$$

where δt is the time step size and u^h is used to show $u(x, t + \delta t)$. To linearize the nonlinear term $u^{h+1} u_x^{h+1}$, we use the linearization form applying in [7,67–69].

$$(u u_x)^{h+1} = u^{h+1} u_x^h + u^h u_x^{h+1} - u^h u_x^h. \tag{3.5}$$

Using the linearized value of $(uu_x)^{h+1}$ in (3.4), we obtain

$$\frac{u^{h+1} - u^h}{\delta t} + \theta\left(\alpha\left(u^{h+1} u_x^h + u^h u_x^{h+1} - u^h u_x^h\right) - \upsilon u_{xx}^{h+1}\right) + (1-\theta)(\alpha u^h u_x^h - \upsilon u_{xx}^h) = 0. \tag{3.6}$$

Rearranging (3.6) and using the Crank–Nicolson method with $\theta = \frac{1}{2}$, we obtain

$$u^{h+1} + \frac{\alpha \delta t}{2}\left(u^{h+1} u_x^h + u^h u_x^{h+1}\right) - \frac{\upsilon \delta t}{2} u_{xx}^{h+1} = u^h + \frac{\delta t}{2} \upsilon u_{xx}^h. \tag{3.7}$$

Employing (2.1), the unknown function u^h can be approximated as

$$u^h(x) \approx \sum_{m=a}^{b-1} \sum_{k=0}^{r-1} \sum_{l=0}^{2^n-1} u_{nl}^{km} \phi_{nl}^{km}(x) = {U_h}^T \Phi(x), \tag{3.8}$$

where U_h is a $N \times 1$ vector. Also using (2.6), we can write

$$u_x^h \approx U_h^T \frac{d}{dx} \Phi(x) = U_h^T D \Phi(x), \tag{3.9}$$

and

$$u_{xx}^h \approx U_h^T \frac{d^2}{dx^2} \Phi(x) = U_h^T D^2 \Phi(x). \tag{3.10}$$

We assume

$$e_1^{h+1}(x) = u^{h+1} u_x^h, \; e_2^{h+1}(x) = u^h u_x^{h+1}. \tag{3.11}$$

Using (2.1), we obtain

$$e_1^{h+1}(x) \approx E_{1_{h+1}}^T \Phi(x), \; e_2^{h+1}(x) \approx E_{2_{h+1}}^T \Phi(x), \tag{3.12}$$

where $E_{i_{h+1}}, i = 1, 2$ are the $N \times 1$ vectors with entries

$$E_{i_{h+1}} = \left[e_{in0}^{0a}, \ldots, e_{in0}^{r-1,a} | \ldots | e_{in,2^n-1}^{0,a}, \ldots, e_{in,2^n-1}^{r-1,a} | \ldots | e_{in0}^{0,b-1}, \ldots, e_{in0}^{r-1,b-1} | \ldots | e_{in,2^n-1}^{0,b-1}, \ldots, e_{in,2^n-1}^{r-1,b-1}\right]^T, \tag{3.13}$$

$$e_{inl}^{km} \approx 2^{-n/2} \sqrt{\frac{\omega_k}{2}} e_i^{h+1}\left(2^{-n}\left(\hat{\tau}_k + l + m2^n\right)\right), \tag{3.14}$$

$$k = 0, \ldots, r-1, \; l = 0, \ldots, 2^n - 1, \; m = a, \ldots, b-1.$$

We can write

$$E_{i_{h+1}} = U_{h+1}^T \Omega_{i_{h+1}}, \; i = 1, 2, \tag{3.15}$$

where $\Omega_{i_{h+1}}, i = 1, 2$ are $N \times N$ known matrices. Now replacing (3.8), (3.10), and (3.15) in (3.7) yields

$$U_{h+1}^T \left(I + \frac{\alpha \delta t}{2} \left(\Omega_{1_{h+1}} + \Omega_{2_{h+1}} \right) - \frac{\upsilon \delta t}{2} D^2 \right) \Phi(x) = U_h^T \left(I + \frac{\upsilon \delta t}{2} D^2 \right) \Phi(x). \tag{3.16}$$

Let

$$M_1 = \left(I + \frac{\alpha \delta t}{2} \left(\Omega_{1_{h+1}} + \Omega_{2_{h+1}} \right) - \frac{\upsilon \delta t}{2} D^2 \right),$$

and

$$M_2 = \left(I + \frac{\upsilon \delta t}{2} D^2 \right),$$

then (3.16) can be written as

$$U_{h+1}^T M_1 \Phi(x) = U_h^T M_2 \Phi(x). \tag{3.17}$$

The entries of vector $\Phi(x)$ are independent, so we obtain

$$U_{h+1}^T M_1 = U_h^T M_2. \tag{3.18}$$

Using (3.3), we have

$$U_{h+1}^T \Phi(a) = g_1((h+1)\delta t), \quad U_{h+1}^T \Phi(b) = g_2((h+1)\delta t). \tag{3.19}$$

Replacing $\Phi(a)$ and $\Phi(b)$ at the first and last columns of M_1, respectively, and using

$$\left\{ U_h^T M_2 \right\}_{1,1} = g_1((h+1)\delta t),$$

and

$$\left\{ U_h^T M_2 \right\}_{1,N} = g_2((h+1)\delta t),$$

we can write

$$U_{h+1}^T \tilde{M}_1 = U_h^T \tilde{M}_2 + F_1, \tag{3.20}$$

where

$$\tilde{M}_2 = \begin{bmatrix} 1 & M_{2_{1,2}} & \cdots & M_{2_{1,N-1}} & 0 \\ 0 & M_{2_{2,2}} & \cdots & M_{2_{2,N-1}} & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & M_{2_{N,2}} & \cdots & M_{2_{N,N-1}} & 1 \end{bmatrix},$$

$$F_1 = [g_1((h+1)\delta t) - U_{h_{1,1}}, 0, \cdots, 0, g_2((h+1)\delta t) - U_{h_{N,1}}]^T.$$

Using (2.1), the function $f(x)$ can be approximated as

$$f(x) \approx F^T \Phi(x). \tag{3.21}$$

Let

$$U_0^T = F^T. \tag{3.22}$$

Equation (3.20) using (3.22) as the starting points gives a linear system of equations, which can be solved to find U_{h+1} in any step $h = 1, 2, \ldots$. So the unknown function $u(x, t_h)$ at any time $t = t_h, h = 1, 2, \ldots$, can be found.

3.2. The mixed finite difference method for equation E2

Consider the KdV–Burgers equation

$$u_t + \alpha u u_x - \upsilon u_{xx} + \mu u_{xxx} = 0, \quad (x,t) \in [a,b] \times [0,T], \tag{3.23}$$

with the initial condition

$$u(x,0) = \tilde{f}(x), \quad x \in [a,b], \tag{3.24}$$

and the boundary conditions

$$u(a,t) = \tilde{g}_1(t), \;\; u(b,t) = \tilde{g}_2(t), \quad t \in [0,T],$$

$$u_x(b,t) = \tilde{h}(t), \quad t \in [0,T]. \tag{3.25}$$

Using the following θ-weighted scheme for KdV–Burgers equation, we obtain

$$\frac{u^{h+1} - u^h}{\delta t} + \theta\left(\alpha u^{h+1}u_x^{h+1} - \upsilon u_{xx}^{h+1} + \mu u_{xxx}^{h+1}\right) + (1-\theta)\left(\alpha u^h u_x^h - \upsilon u_{xx}^h + \mu u_{xxx}^h\right) = 0, \tag{3.26}$$

where δt is the time step size and u^h is used to show $u(x, t+\delta t)$. To linearize the nonlinear term $u^{h+1}u_x^{h+1}$, we use the linearization form described in Subsection 3.1. Rearranging (3.26) and using the Crank–Nicolson method with $\theta = \frac{1}{2}$, we obtain

$$u^{h+1} + \frac{\alpha\delta t}{2}\left(u^{h+1}u_x^h + u^h u_x^{h+1}\right) - \frac{\upsilon\delta t}{2}u_{xx}^{h+1} + \frac{\mu\delta t}{2}u_{xxx}^{h+1} = u^h + \frac{\delta t}{2}\upsilon u_{xx}^h - \frac{\mu\delta t}{2}u_{xxx}^h. \tag{3.27}$$

Using (3.8)–(3.12) and (3.15), we obtain

$$U_{h+1}^T\left(I + \frac{\alpha\delta t}{2}\left(\Omega_{1_{h+1}} + \Omega_{2_{h+1}}\right) - \frac{\upsilon\delta t}{2}D^2 + \frac{\mu\delta t}{2}D^3\right)\Phi(x) = U_h^T\left(I + \frac{\upsilon\delta t}{2}D^2 - \frac{\mu\delta t}{2}D^3\right)\Phi(x). \tag{3.28}$$

By replacing $M_3 = \left(I + \frac{\alpha\delta t}{2}\left(\Omega_{1_{h+1}} + \Omega_{2_{h+1}}\right) - \frac{\upsilon\delta t}{2}D^2 + \frac{\mu\delta t}{2}D^3\right)$, and $M_4 = \left(I + \frac{\upsilon\delta t}{2}D^2 - \frac{\mu\delta t}{2}D^3\right)$, we obtain

$$U_{h+1}^T M_3 \Phi(x) = U_h^T M_4 \Phi(x). \tag{3.29}$$

The entries of vector $\Phi(x)$ are independent, so we obtain

$$U_{h+1}^T M_3 = U_h^T M_4. \tag{3.30}$$

Using (3.25), we have

$$\begin{aligned} U_{h+1}^T\Phi(a) &= \tilde{g}_1((h+1)\delta t),\\ U_{h+1}^T\Phi(b) &= \tilde{g}_2((h+1)\delta t),\\ U_{h+1}^T D\Phi(b) &= \tilde{h}((h+1)\delta t). \end{aligned} \tag{3.31}$$

Replacing $\Phi(a)$, $D\Phi(b)$, and $\Phi(b)$ at the first, second, and last columns of M_3, respectively, and using the relations

$$\left\{U_h^T M_4\right\}_{1,1} = \tilde{g}_1((h+1)\delta t),$$
$$\left\{U_h^T M_4\right\}_{1,N} = \tilde{g}_2((h+1)\delta t),$$

and

$$\left\{U_h^T M_4\right\}_{1,2} = \tilde{h}((h+1)\delta t),$$

we can write

$$U_{h+1}^T\tilde{M}_3 = U_h^T\tilde{M}_4 + F_3, \tag{3.32}$$

where

$$\tilde{M}_4 = \begin{bmatrix} 1 & 0 & M_{4_{1,2}} & \cdots & M_{4_{1,N-1}} & 0 \\ 0 & 1 & M_{4_{2,2}} & \cdots & M_{4_{2,N-1}} & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & M_{4_{N,2}} & \cdots & M_{4_{N,N-1}} & 1 \end{bmatrix},$$

$$F_3 = \left[\tilde{g}_1((h+1)\delta t) - U_{h_{1,1}}, \tilde{h}((h+1)\delta t) - U_{h_{2,1}}, 0, \cdots, 0, \tilde{g}_2((h+1)\delta t) - U_{h_{N,1}}\right]^T.$$

Now using (3.24), $\tilde{f}(x)$ can be approximated as

$$\tilde{f}(x) \approx \tilde{F}^T \Phi(x). \tag{3.33}$$

Let

$$U_0^T = \tilde{F}^T. \tag{3.34}$$

Equation (3.32) using (3.34) as the starting points gives a linear system of equations, which can be solved to find U_{h+1} in any step $h = 1, 2, \ldots$. So the unknown function $u(x, t_h)$ at any time $t = t_h, h = 1, 2, \ldots$, can be found.

3.3. *The mixed finite difference method for coupled Burgers equation*

Consider the nonlinear system of Burgers equations

$$\begin{cases} u_t - u_{xx} + 2uu_x + \alpha(uv)_x = 0, \\ v_t - v_{xx} + 2vv_x + \beta(uv)_x = 0, \end{cases} \tag{3.35}$$

with the boundary conditions

$$\begin{cases} u(a,t) = f_1(t) \quad v(a,t) = g_1(t) \quad t > 0, \\ u(b,t) = f_2(t) \quad v(b,t) = g_2(t) \quad t > 0, \end{cases} \tag{3.36}$$

and initial conditions

$$\{ \; u(x,0) = f(x) \quad v(x,0) = g(x) \quad x \in [a,b], \tag{3.37}$$

where α and β are real parameters. We use θ-weighted scheme again for these equations as

$$\begin{cases} \frac{u^{h+1}-u^h}{\delta t} + \theta\left(-u_{xx}^{h+1} + 2u^{h+1}u_x^{h+1}\right) + (1-\theta)\left(-u_{xx}^h + 2u^h u_x^h\right) + \alpha\left(u^h v_x^h + u_x^h v^h\right) = 0, \\ \frac{v^{h+1}-v^h}{\delta t} + \theta\left(-v_{xx}^{h+1} + 2v^{h+1}v_x^{h+1}\right) + (1-\theta)\left(-v_{xx}^h + 2v^h v_x^h\right) + \beta\left(u^h v_x^h + u_x^h v^h\right) = 0. \end{cases} \tag{3.38}$$

Using linearized forms of nonlinear terms $u^{h+1}u_x^{h+1}$ and $v^{h+1}v_x^{h+1}$ and also using Crank–Nicolson method $(\theta = \frac{1}{2})$, and by rearranging (3.38), we obtain

$$\begin{cases} u^{h+1} - \frac{\delta t}{2}u_{xx}^{h+1} + \delta t\left(u^h u_x^{h+1} + u_x^h u^{h+1}\right) = u^h + \frac{\delta t}{2}u_{xx}^h - \alpha\delta t\left(u_x^h v^h + u^h v_x^h\right), \\ v^{h+1} - \frac{\delta t}{2}v_{xx}^{h+1} + \delta t\left(v^h v_x^{h+1} + v_x^h v^{h+1}\right) = v^h + \frac{\delta t}{2}v_{xx}^h - \beta\delta t\left(u_x^h v^h + u^h v_x^h\right). \end{cases} \tag{3.39}$$

Using (2.1), the unknown function v^h can be approximated as

$$v^h(x) \approx \sum_{m=a}^{b-1}\sum_{k=0}^{r-1}\sum_{l=0}^{2^n-1} v_{nl}^{km}\phi_{nl}^{km}(x) = V_h{}^T\Phi(x), \tag{3.40}$$

where V_h is a $N \times 1$ vector. Also using (2.6), we can write

$$v_x^h \approx V_h^T \frac{d}{dx}\Phi(x) = V_h^T D\Phi(x), \tag{3.41}$$

and

$$v_{xx}^h \approx V_h^T \frac{d^2 2}{dx^2}\Phi(x) = V_h^T D^2\Phi(x). \tag{3.42}$$

At this work, we assume

$$e_3^{h+1}(x) = v^{h+1}v_x^h,\ e_4^{h+1}(x) = v^h v_x^{h+1}, \tag{3.43}$$

$$e_5^{h+1}(x) = v^h u_x^h,\ e_6^{h+1}(x) = u^h v_x^h. \tag{3.44}$$

Using (2.1) in (3.44), we obtain

$$e_3^{h+1}(x) \approx {E_{3_{h+1}}}^T\Phi(x),\ e_4^{h+1}(x) \approx {E_{4_{h+1}}}^T\Phi(x), \tag{3.45}$$

$$e_5^{h+1}(x) \approx {E_{5_{h+1}}}^T\Phi(x),\ e_6^{h+1}(x) \approx {E_{6_{h+1}}}^T\Phi(x), \tag{3.46}$$

where $E_{i_{h+1}}, i = 3, \ldots, 6$ are the $N \times 1$ vectors with entries the same as (3.14).

We can write

$$E_{i_{h+1}} = U_{h+1}^T \Omega_{i_{h+1}},\ i = 3, \ldots, 6, \tag{3.47}$$

where $\Omega_{i_{h+1}}$ are the $N \times N$ matrices with known entries. Now by replacing (3.8)–(3.10), (3.40)–(3.42), and (3.15), (3.47) in (3.39), we have

$$\begin{cases} {U_{h+1}}^T \left(I - \frac{\delta t}{2}D^2 + \delta t\left(\Omega_{1_{h+1}} + \Omega_{2_{h+1}}\right)\right)\Phi(x) = {U_h}^T\left(I + \frac{\delta t}{2}D^2 - \alpha\delta t\left(\Omega_{5_{h+1}} + \Omega_{6_{h+1}}\right)\right)\Phi(x), \\ V_{h+1}^T \left(I - \frac{\delta t}{2}D^2 + \delta t\left(\Omega_{3_{h+1}} + \Omega_{4_{h+1}}\right)\right)\Phi(x) = V_h^T\left(I + \frac{\delta t}{2}D^2 - \beta\delta t\left(\Omega_{5_{h+1}} + \Omega_{6_{h+1}}\right)\right)\Phi(x). \end{cases} \tag{3.48}$$

Let

$$\begin{aligned} M_5 &= I - \frac{\delta t}{2}D^2 + \delta t\left(\Omega_{1_{h+1}} + \Omega_{2_{h+1}}\right), \\ M_6 &= I + \frac{\delta t}{2}D^2 - \alpha\delta t\left(\Omega_{5_{h+1}} + \Omega_{6_{h+1}}\right), \\ M_7 &= I - \frac{\delta t}{2}D^2 + \delta t\left(\Omega_{3_{h+1}} + \Omega_{4_{h+1}}\right), \\ M_8 &= I + \frac{\delta t}{2}D^2 - \beta\delta t\left(\Omega_{5_{h+1}} + \Omega_{6_{h+1}}\right). \end{aligned}$$

Using (3.48) and because of the independency in the entries of vector $\Phi(x)$, we obtain

$$\begin{cases} {U_{h+1}}^T M_5 = {U_h}^T M_6, \\ V_{h+1}^T M_7 = V_h^T M_8. \end{cases} \tag{3.49}$$

Replacing $\Phi(a)$ and $\Phi(b)$ in the first and last columns of M_5 and M_7, respectively, and regarding the relations

$$\begin{cases} \{U_h{}_6^M\}_{1,1} = f_1((h+1)\delta t) \quad \{U_h{}_8^M\}_{1,1} = g_1((h+1)\delta t), \\ \{U_h{}_6^M\}_{1,N} = f_2((h+1)\delta t) \quad \{U_h{}_8^M\}_{1,N} = g_2((h+1)\delta t), \end{cases}$$

we obtain

$$\begin{cases} {U_{h+1}}^T \tilde{M}_5 = {U_h}^T \tilde{M}_6 + F_5, \\ V_{h+1}^T \tilde{M}_7 = V_h^T \tilde{M}_8 + F_7, \end{cases} \tag{3.50}$$

where

$$\tilde{M}_6 = \begin{bmatrix} 1 & M_{6_{1,2}} & \cdots & M_{6_{1,N-1}} & 0 \\ 0 & M_{6_{2,2}} & \cdots & M_{6_{2,N-1}} & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & M_{6_{N,2}} & \cdots & M_{6_{N,N-1}} & 1 \end{bmatrix},\quad \tilde{M}_8 = \begin{bmatrix} 1 & M_{8_{1,2}} & \cdots & M_{8_{1,N-1}} & 0 \\ 0 & M_{8_{2,2}} & \cdots & M_{8_{2,N-1}} & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & M_{8_{N,2}} & \cdots & M_{8_{N,N-1}} & 1 \end{bmatrix},$$

$$F_5 = [f_1((h+1)\delta t) - U_{h_{1,1}}, 0, \cdots, 0, f_2((h+1)\delta t) - U_{h_{N,1}}]^T,$$

and

$$F_7 = [g_1((h+1)\delta t) - U_{h_{1,1}}, 0, \cdots, 0, g_2((h+1)\delta t) - U_{h_{N,1}}]^T.$$

Using (3.37), the functions $f(x)$ and $g(x)$ can be approximated as

$$f(x) \approx F^T \Phi(x), \quad g(x) \approx G^T \Phi(x). \tag{3.51}$$

Suppose

$$U_0^T = F^T, \quad V_0^T = G^T. \tag{3.52}$$

Equation (3.50) using (3.52) as the starting points gives a linear system of equations, which can be solved to find U_{h+1} and V_{h+1} in any step $h = 1, 2, \ldots$. So the unknown functions $u(x, t_h)$ and $v(x, t_h)$ in any time $t = t_h, h = 1, 2, \ldots$, can be found.

3.4. *The stability analysis*

In this section, we present the stability analysis of scheme (3.7), (3.27), and (3.39) by using the ISFs of the amplification matrix. Let $\hat{u}$ and $\hat{v}$ be the exact solutions and belong to the $L^2[a, b]$, $\{u, v\}$, and $\{u^*, v^*\}$ be the projection solutions and numerical solutions of (3.1), (3.23), and (3.35) in nth space, respectively, then the error vector $\epsilon_i^h, i = 1, 2$ is defined by

$$\epsilon_1^h = u^h - u^{*h}, \quad \epsilon_2^h = v^h - v^{*h}.$$

Regarding the relations $u^h = U^T \Phi(x)$ and $v^h = V^T \Phi(x)$, we obtain

$$\|\epsilon_1^h\| = \|u^h - u^{*h}\| = \|U^T \Phi(x) - U^{*T} \Phi(x)\| = \|U^T - U^{*T}\|, \quad \|\Phi(x)\| = 1. \tag{3.53}$$

For the stability of the numerical scheme, we must have $\epsilon_i^h \to 0$ as $h \to \infty$. This implies $\|U^T - U^{*T}\| \to 0$.

Let the matrices $\{M_i, i = 1, 3, 5, 7\}$ be invertible. Because it is not possible to show these matrices are invertible in general [7], the minimum eigenvalues of the matrices $\{M_i, i = 1, 3, 5, 7\}$ for the presented examples are recorded in Tables 4, 7, 11, and 14. Thus, we have

$$U_{h+1}^T = U_h^T \tilde{M}_{i+1} \tilde{M}_i^{-1} + F_i \tilde{M}_i^{-1}, \quad i = 1, 3, 5, \tag{3.54}$$

and for the coupled Burgers equation we have also

$$V_{h+1}^T = V_h^T \tilde{M}_8 \tilde{M}_7^{-1} + F_7 \tilde{M}_7^{-1}. \tag{3.55}$$

In the $(h+1)$th step, we have

$$U_{h+1}^T - {U_{h+1}^*}^T = \left(U_h^T - U_h^{*T}\right) \tilde{M}_{i+1} \tilde{M}_i^{-1}, \quad i = 1, 3, 5, \tag{3.56}$$

$$V_{h+1}^T - {V_{h+1}^*}^T = \left(V_h^T - V_h^{*T}\right) M_8 M_7^{-1}. \tag{3.57}$$

Using (3.56) and (3.57), we obtain

$$\|U_{h+1}^T - {U_{h+1}^*}^T\| \le \|U_h^T - U_h^{*T}\| \|\tilde{M}_{i+1} \tilde{M}_i^{-1}\|, \quad i = 1, 3, 5, \tag{3.58}$$

$$\|V_{h+1}^T - {V_{h+1}^*}^T\| \le \|V_h^T - V_h^{*T}\| \|\tilde{M}_8 \tilde{M}_7^{-1}\|. \tag{3.59}$$

The necessary and sufficient conditions to obtain $\|U_{h+1}^T - {U_{h+1}^*}^T\| \to 0$ and $\|V_{h+1}^T - {V_{h+1}^*}^T\| \to 0$, is $\rho_i < 1$ where $\{\rho_i, i = 1, 3, 5\}$ are the spectral radius of the matrices $\tilde{M}_{i+1} \tilde{M}_i^{-1}, i = 1, 3, 5$, and ρ_7 is the spectral radius of the matrix $\tilde{M}_8 \tilde{M}_7^{-1}$. These results show that the approximate solution of equations approaches to the projection solution of these equations.

Lemma 3.1
Suppose the function $f : [0, 1] \to R$ is r times continuously differentiable. Then $P_n^r f$ approximates f with the mean error bounded as follows [70]:

$$\|P_n^r f - f\| \le 2^{-nr} \frac{2}{4^r r!} \sup_{x \in [0,1]} |f^{(r)}(x)|.$$

This lemma implies that the projection solution is close to the exact solution.

4. Numerical experiments

In this section, some numerical examples are presented to illustrate the validity and the merits of the new technique. We report L_∞ and the L_2 errors of the solution that are defined as

$$L_\infty = \max_{0\le i\le 10} |u_i - \tilde{u}_i|,$$

and

$$L_2 = \left(\int_0^1 |u_i - \tilde{u}_i|^2 dx \right)^{\frac{1}{2}},$$

where u_i and $\tilde{u}_i$ are the exact and computed values of the solution u at the points $t_i = \frac{i}{10}, i = 0, \cdots, 10$, respectively. In all examples, the initial and boundary conditions are computed from the exact solutions. Note that all test problems are taken from the literature.

Example 4.1
Consider the 1-D Burgers equation:

$$u_t + \alpha u u_x - \upsilon u_{xx} = 0, \quad (x,t) \in D \times [0,1], \tag{4.1}$$

with the solitary wave solutions [6]

$$u(x,t) = \frac{c}{\alpha} + \left(\frac{2\upsilon}{\alpha}\right) \tanh(x - ct), \tag{4.2}$$

in the region $D = \{x : 0 < x < 1\}$; α and υ are arbitrary constants.

L_∞ and L_2 errors for various values of α and υ with $r = 5, n = 2$ and $c = 0.01$, are reported in Table I. Table II shows the spectral radius of the matrix $\tilde{M}_2\tilde{M}_1^{-1}$ for different values of t. Also in Table III, we compare L_2 error of Example 4.1 with the results of [6] for $r = 5, n = 2$ and $c = 0.1$. Table IV, shows the minimum eigenvalues of matrices $\tilde{M}_1$ for various values of α, υ and different values of t. Figure 1 shows the absolute error and compares the exact and approximate solutions of (4.1) with $\alpha = 1, \upsilon = 0.0001$ and $c = 0.01$ for $r = 3, n = 2$, $a = -3$ and $b = 3$ in $t = 1$.

Example 4.2
Consider the KdV–Burgers equation

$$u_t + \epsilon u u_x - \upsilon u_{xx} + \mu u_{xxx} = 0, \tag{4.3}$$

Table I. L_∞ and L_2 errors for various values of α and υ for Example 4.1.

			L_2 Error			L_∞ Error	
α	υ	$t = 0.3$	$t = 0.6$	$t = 1.0$	$t = 0.3$	$t = 0.6$	$t = 1.0$
0.1	0.01	7.14×10^{-4}	1.39×10^{-3}	2.24×10^{-3}	9.11×10^{-4}	1.81×10^{-3}	2.99×10^{-3}
	0.001	7.88×10^{-6}	1.53×10^{-5}	2.49×10^{-5}	9.26×10^{-6}	1.85×10^{-5}	3.26×10^{-5}
	0.0001	8.38×10^{-8}	1.70×10^{-7}	2.89×10^{-7}	7.79×10^{-8}	2.41×10^{-7}	4.13×10^{-7}
1	0.01	7.14×10^{-5}	1.39×10^{-4}	2.24×10^{-4}	9.11×10^{-5}	1.81×10^{-4}	2.99×10^{-4}
	0.001	7.88×10^{-7}	1.56×10^{-6}	2.49×10^{-6}	1.09×10^{-6}	2.06×10^{-6}	3.26×10^{-6}
	0.0001	8.36×10^{-9}	1.70×10^{-8}	2.89×10^{-8}	1.18×10^{-8}	2.41×10^{-8}	4.13×10^{-8}

Table II. Spectral radius of the matrix $\tilde{M}_2\tilde{M}_1^{-1}$ for Example 4.1.

α	υ	$t = 0.3$	$t = 0.6$	$t = 1.0$
0.1	0.01	0.9990389161	0.9990393784	0.9990399715
	0.001	0.9998268459	0.9998269033	0.9998269795
	0.0001	0.9999526577	0.9999526560	0.9999526537
1	0.01	0.9990389161	0.9990393784	0.9990399715
	0.001	0.9998268459	0.9998269033	0.9998269795
	0.0001	0.9999526577	0.9999526560	0.9999526537

Table III. Comparison of L_2 norm of Example 4.1 with results from [6].

		Mixed finite difference and Galerkin methods			[6]		
α	υ	$t=0.1$	$t=0.2$	$t=0.25$	$t=0.1$	$t=0.2$	$t=0.25$
0.1	0.01	3.06×10^{-4}	6.12×10^{-4}	7.64×10^{-4}	3.06×10^{-4}	6.11×10^{-4}	7.62×10^{-4}
	0.001	3.08×10^{-6}	6.37×10^{-6}	7.98×10^{-6}	3.10×10^{-6}	6.32×10^{-6}	7.99×10^{-6}
	0.0001	3.40×10^{-8}	8.80×10^{-8}	1.34×10^{-7}	7.15×10^{-7}	1.31×10^{-6}	1.67×10^{-6}
1	0.01	3.06×10^{-5}	6.12×10^{-5}	7.64×10^{-5}	3.06×10^{-5}	6.11×10^{-5}	7.62×10^{-5}
	0.001	3.08×10^{-7}	6.37×10^{-7}	7.98×10^{-7}	3.10×10^{-7}	6.32×10^{-7}	7.99×10^{-7}
	0.0001	3.40×10^{-9}	8.80×10^{-9}	1.34×10^{-8}	2.24×10^{-8}	5.22×10^{-8}	8.94×10^{-8}

Table IV. Minimum eigenvalue of the matrix $\tilde{M}_1$, for Example 4.1.

α	υ	$t=0.3$	$t=0.6$	$t=1.0$
0.1	0.01	0.9654916	0.9654545	0.9654076
	0.001	0.5613544	0.5613486	0.5613407
	0.0001	0.4614543	0.4614541	0.4614538
1	0.01	0.9654916	0.9654545	0.9654076
	0.001	0.5613544	0.5613486	0.5613407
	0.0001	0.4614543	0.4614541	0.4614538

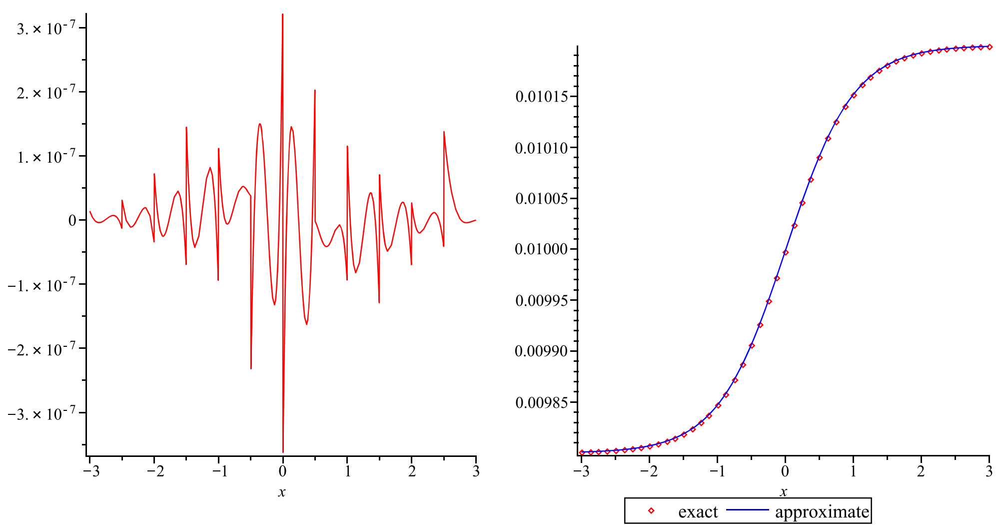

Figure 1. Absolute error (left) and comparing the exact and approximate solutions (right) for Example 4.1 with $r = 3$, $n = 2$, $\alpha = 1$, $\upsilon = 0.0001$, $c = 0.01$ in $[-3, 3]$ and $t = 1$.

with the solitary wave solutions [6]

$$u(x,t) = A\left[9 - 6\tanh(B(x-Ct)) - 3\tanh^2(B(x-Ct))\right], \tag{4.4}$$

in the region $D = \{x : 0 < x < 1\}$; $A = \frac{\upsilon^2}{25\epsilon\mu}$, $B = \frac{\upsilon}{10\mu}$, $C = \frac{6\upsilon^2}{25\mu}$, ϵ, υ and μ are arbitrary constants.

Table V, shows L_∞ and L_2 errors for $r = 5$, $n = 2$, and various values of μ, υ, and ϵ. Table VI contains the spectral radius of the matrix $\tilde{M}_4\tilde{M}_3^{-1}$ for different values of t. The minimum eigenvalue of matrices $\tilde{M}_3$ for various values of ϵ, υ, μ, and different values of t, is reported in Table VII. Figure 2 demonstrates the plot of absolute error and the plot of exact and approximate solutions.

Table V. L_∞ and L_2 errors for various values of ϵ and υ for Example 4.2.

			L_2*Error*		L_∞*Error*		L_∞*Error* [6]	
ϵ	υ	μ	$t=0.3$	$t=0.9$	$t=0.3$	$t=0.9$	$t=0.3$	$t=0.9$
1	0.1	0.1	1.28×10^{-12}	2.39×10^{-12}	3.71×10^{-12}	7.51×10^{-12}	8.20×10^{-8}	2.26×10^{-7}
	0.1	1.0	7.66×10^{-19}	1.63×10^{-18}	2.56×10^{-19}	4.83×10^{-19}	1.07×10^{-8}	6.31×10^{-8}
0.1	0.1	0.1	1.28×10^{-11}	2.39×10^{-11}	3.71×10^{-11}	7.51×10^{-11}	8.27×10^{-7}	2.27×10^{-6}
	0.1	1.0	2.56×10^{-18}	4.83×10^{-18}	3.00×10^{-18}	1.63×10^{-17}	2.01×10^{-7}	5.92×10^{-7}

Table VI. Spectral radius of the matrix $\tilde{M}_4\tilde{M}_3^{-1}$ for Example 4.2.

ϵ	υ	μ	$t=0.3$	$t=0.9$
1	0.1	0.1	0.9997969093	0.9997968536
	0.1	1.0	0.9999031714	0.9999031526
0.1	0.1	0.1	0.9997969093	0.9997968549
	0.1	1.0	0.9999031639	0.9999031526

Table VII. Minimum eigenvalue of the matrix $\tilde{M}_3$ for Example 4.2.

ϵ	υ	μ	$t=0.3$	$t=0.9$
1	0.1	0.1	1.03610780	1.036107967
	0.1	1.0	1.36545996	1.36545999
0.1	0.1	0.1	1.03610797	1.03610803
	0.1	1.0	1.36545999	1.36545999

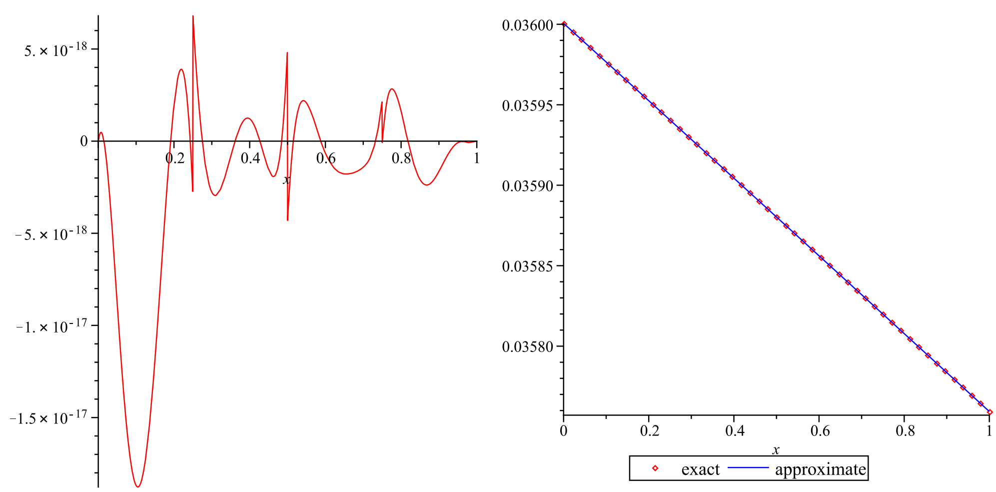

Figure 2. Absolute error (left) and comparing the exact and approximate solutions (right) for Example 4.2 with $\upsilon=0.1$, $\epsilon=0.1$, and $\mu=1$ using $r=5$ and $n=2$ in $t=1$.

Example 4.3

Consider the coupled Burgers Equations (1.7) which the boundary and initial values are given by the exact solution. For this example, exact solutions are given [7] as

$$\begin{cases} u(x,t)=a_0-2A\left(\frac{2\alpha-1}{4\alpha\beta-1}\right)\tanh\left[A(x-2At)\right], \\ v(x,t)=a_0\frac{2\beta-1}{2\alpha-1}-2A\left(\frac{2\alpha-1}{4\alpha\beta-1}\right)\tanh\left[A(x-2At)\right], \\ a<x<b, \end{cases}$$

with $A=\frac{1}{2}\left(\frac{a_0(4\alpha\beta-1)}{2\alpha-1}\right)$, a_0, α and β are arbitrary constants.

Table VIII, shows the L_∞ errors with $a_0 := 0.05$, $\alpha = 1.0$ and $\beta = 0.3$ at time levels $t = 0.5$ and 1 and compares the L_∞ error of Example 4.3 with results from [6, 7]. Table IX, shows the spectral radius of the matrix $\tilde{M}_6\tilde{M}_5^{-1}$ for different values of t. In Table X, we compare L_∞ error of Example 4.3 with results from [6] with $a_0 := 0.05$, $\alpha = 0.1$ and $\beta = 0.3$ at time levels $t = 0.5$ and 1. Table XI, reports the minimum eigenvalue of matrices $\tilde{M}_5$ for different values of t. Figure 3 demonstrates the plot of absolute errors. Figure 4 shows the exact and approximation solutions at $t = 1$.

Table VIII. Comparison of norm infinity of Example 4.3 with results from [6, 7] for $a_0 := 0.05$, $\alpha = 1.0$, $\beta = 0.3$, $a = 0$ and $b = 1$.

	Ł$_\infty$ *Error* for mixed finite difference and Galerkin methods		L_∞ *Error* for method [6]		L_∞ *Error* for method [7]	
	$t = 0.5$	$t = 1.0$	$t = 0.5$	$t = 1.0$	$t = 0.5$	$t = 1.0$
u	2.47×10^{-6}	2.49×10^{-6}	8.81×10^{-6}	8.82×10^{-6}	3.70×10^{-6}	3.73×10^{-6}
v	4.04×10^{-7}	4.07×10^{-7}	2.86×10^{-6}	2.86×10^{-6}	8.91×10^{-7}	8.98×10^{-7}

Table IX. Spectral radius of the matrix $\tilde{M}_6\tilde{M}_5^{-1}$, for Example 4.3.

	$t = 0.5$	$t = 1.0$
u	0.9128635979	0.9128635921
v	0.9129552052	0.9129551996

Table X. Comparison of norm infinity of Example 4.3 with results from method presented in [6] for $a_0 := 0.05$, $\alpha = 0.1$, $\beta = 0.3$, $a = -10$, and $b = 10$.

	Mixed finite difference and Galerkin methods		Method [6]	
	$t = 0.5$	$t = 1.0$	$t = 0.5$	$t = 1.0$
u	4.23×10^{-5}	8.28×10^{-5}	4.38×10^{-5}	8.66×10^{-5}
v	2.51×10^{-5}	4.78×10^{-5}	4.99×10^{-5}	9.92×10^{-5}

Table XI. Minimum eigenvalues of the matrices $\tilde{M}_5$ and $\tilde{M}_7$ for Example 4.3.

	$t = 0.5$	$t = 1.0$
u	1.04098615	1.04098317
v	1.04165094	1.04165095

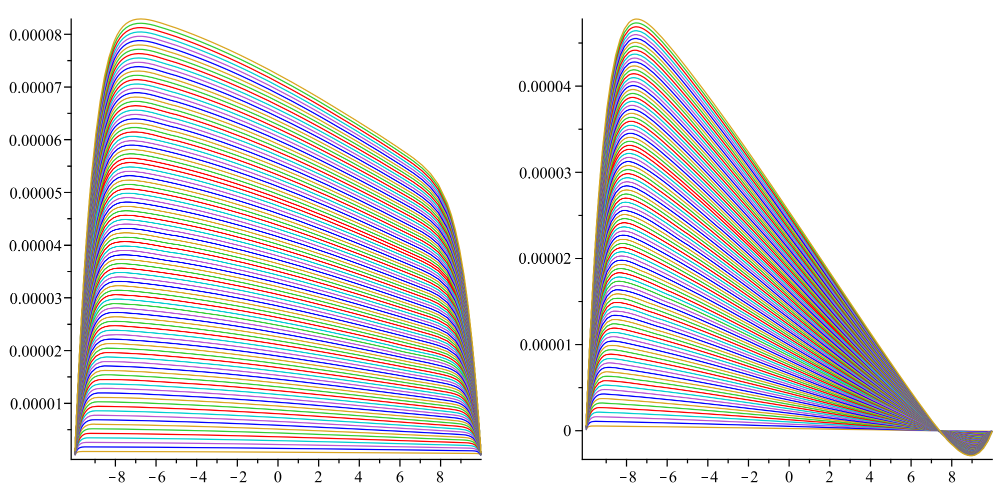

Figure 3. Absolute errors for mixed finite difference and Galerkin methods for Example 4.3 with $a_0 = 0.05$, $\alpha = 0.1$, and $\beta = 0.3$ with $r = 4$, $n = 2$. left (u) and right (v).

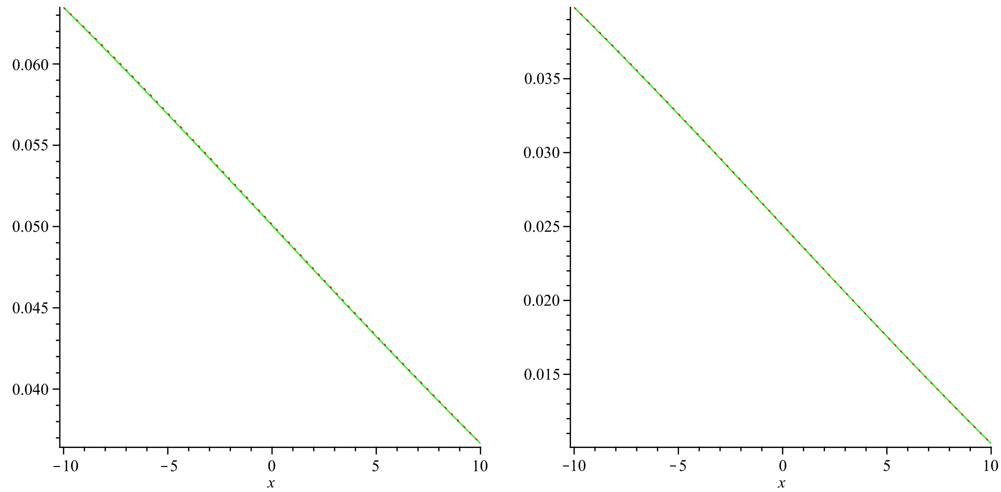

Figure 4. Exact and approximation solutions for Example 4.3 $a_0 = 0.05$, $\alpha = 0.1$, and $\beta = 0.3$ with $r = 4$, $n = 2$. left (u) and right (v).

Table XII. Comparison of the numerical results for $\upsilon = 1$ for Example 4.4.

Method	t	$x = 0.1$	$x = 0.3$	$x = 0.5$	$x = 0.7$	$x = 0.9$
IFDM	0.05	0.17832	0.47658	0.60984	0.51165	0.20006
BEM		0.17759	0.47531	0.60851	0.51050	0.19933
MFDGM		0.17788	0.47555	0.60969	0.51105	0.19990
Exact		0.17803	0.47586	0.60907	0.51113	0.19989
IFDM	0.1	0.11009	0.29335	0.37342	0.31144	0.12128
BEM		0.10931	0.29124	0.37070	0.30911	0.12031
MFDGM		0.10946	0.29167	0.37169	0.30966	0.12059
Exact		0.10954	0.29190	0.37158	0.30991	0.12069
IFDM	0.2	0.04273	0.11276	0.14120	0.11574	0.04457
BEM		0.04220	0.11044	0.13809	0.11322	0.04391
MFDGM		0.04187	0.11046	0.13828	0.11328	0.04361
Exact		0.04193	0.11062	0.13847	0.11347	0.04369

IFDM, implicit finite difference method; BEM, boundary element method; MFDGM, mixed finite difference and Galerkin methods.

Table XIII. Comparison of the numerical results for Example 4.4 with $\upsilon = 0.1$.

Method	t	$x = 0.1$	$x = 0.3$	$x = 0.5$	$x = 0.7$	$x = 0.9$
IFDM	0.5	0.11048	0.32367	0.50447	0.57664	0.30912
BEM		0.10986	0.32191	0.50240	0.57514	0.30779
MFDGM		0.11379	0.32441	0.51059	0.57666	0.30964
Exact		0.10992	0.32219	0.50279	0.57585	0.30935
IFDM	1	0.06689	0.19445	0.29448	0.31107	0.14769
BEM		0.06644	0.19263	0.29139	0.30711	0.14507
MFDGM		0.06802	0.19503	0.29717	0.30953	0.14697
Exact		0.06632	0.19279	0.29192	0.30809	0.14607
IFDM	2	0.02909	0.08044	0.10939	0.09838	0.04037
BEM		0.02913	0.07951	0.10770	0.09663	0.03976
MFDGM		0.02904	0.08020	0.10956	0.09828	0.04119
Exact		0.02876	0.07946	0.10789	0.09685	0.03969

IFDM, implicit finite difference method; BEM, boundary element method; MFDGM, mixed finite difference and Galerkin methods.

Table XIV. Minimum eigenvalue of the matrix $\tilde{M}_1$ for Example 4.4.

υ	$t=0.05$	$t=0.1$	$t=0.2$
1	0.00409012	0.00407345	0.00405001
	$t=0.5$	$t=1$	$t=2$
0.1	0.03465394	0.01916844	0.03175529

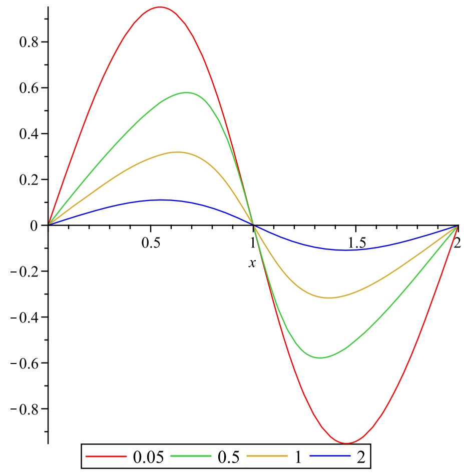

Figure 5. Numerical solution at different times for $\upsilon = 0.1$ for Example 4.4.

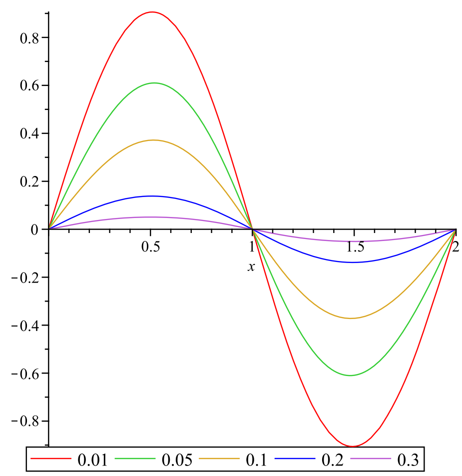

Figure 6. Numerical solution at different times for $\upsilon = 1$ for Example 4.4.

Example 4.4

In this example, we solve the 1-D Burgers equation with different initial and boundary conditions as

$$u(x,0) = sin(\pi x), \quad 0 < x < 2,$$

$$u(0,t) = u(1,t) = 0, \quad t > 0.$$

The exact solution of this equation is

$$u(x,t) = \frac{2\pi\upsilon \sum_{n=1}^{\infty} a_n \exp(-n^2\pi^2\upsilon t) n \sin(n\pi x)}{a_0 + \sum_{n=1}^{\infty} a_n \exp(-n^2\pi^2\upsilon t)\cos(n\pi x)},$$

where the Fourier coefficients are

$$a_0 = \int_0^2 \exp\left\{-(2\pi\upsilon)^{-1}(1-\cos(\pi x))\right\} dx,$$

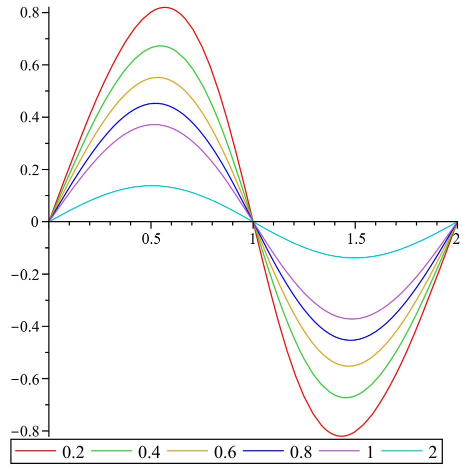

Figure 7. Numerical solution at time 0.1 and different $\upsilon = (0.2, 0.4, 0.6, 0.8, 1, 2)$ for Example 4.4.

and

$$a_n = 2\int_0^2 \exp\{-(2\pi\upsilon)^{-1}(1-\cos(\pi x))\}cos(n\pi x)dx, \quad (n = 1, 2, \cdots).$$

Comparisons are made with exact and numerical solutions of several existing numerical schemes, which are fully implicit finite difference method [71], the mixed finite difference and boundary element methods [72]. The numerical results are presented in Tables XII and XIII for different times and different υ coefficient. In Table XIV, we show the minimum eigenvalue of matrix $\tilde{M}_1$ for different values of υ and t. When the υ is fixed, the numerical solutions at different times are plotted in Figures 5 and 6. We also fix the time and show the numerical simulation with different values of υ in Figure 7. It can be seen that the dissipation effect increases with increasing υ. In all calculations related to the figures, the interval [0, 2] is divided into 200 cells equally.

5. Conclusion

In this article, the 1-D Burgers and KdV–Burgers and the coupled Burgers equations are studied. A numerical method is proposed to find their solutions. This hybrid method uses the finite difference scheme and Galerkin technique based on the interpolating scaling functions (ISFs). The stability of the technique is discussed in some cases. The new procedure was tested on several examples taken from the literature. Numerical simulations are reported to demonstrate the usefulness of the new method proposed in the current work.

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