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Unconditional ideals in Banach spaces

by

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Abstract. We show that a Banach space with separable dual can be renormed to satisfy hereditarily an "almost" optimal uniform smoothness condition. The optimal condition occurs when the canonical decomposition $X^{***} = X^{\perp} \oplus X^*$ is unconditional. Motivated by this result, we define a subspace X of a Banach space Y to be an h-ideal (resp. a u-ideal) if there is an hermitian projection P (resp. a projection P with $\|I - 2P\| = 1$) on Y^* with kernel $X^{\perp}$. We undertake a general study of h-ideals and u-ideals. For example we show that if a separable Banach space X is an h-ideal in X^{**} then X has the complex form of Pełczyński's property (u) with constant one and the Baire-one functions $Ba(X)$ in X^{**} are complemented by an hermitian projection; the converse holds under a compatibility condition which is shown to be necessary. We relate these ideas to the more familiar notion of an M-ideal, and to Banach lattices.

We further investigate when, for a separable Banach space X, the ideal of compact operators $\mathcal{K}(X)$ is a u-ideal or an h-ideal in $\mathcal{L}(X)$ or $\mathcal{K}(X)^{**}$. For example, we show that $\mathcal{K}(X)$ is an h-ideal in $\mathcal{K}(X)^{**}$ if and only if X has the "unconditional compact approximation property" and X is an M-ideal in X^{**}.

1. Introduction. If X is a subspace of a Banach space Y we will say that X is a *summand* of Y if it is the range of a contractive projection; we will say that X is an *ideal* in Y if $X^{\perp}$ is the kernel of a contractive projection on Y^* (this differs from the terminology in [8]). A simple example is that X is always an ideal in its bidual X^{**}. It can be shown that X is an ideal in Y if it is "locally" a summand; more precisely, X is an ideal in Y if and only if for every finite-dimensional subspace F of Y and for every $\varepsilon > 0$, there exists an operator $S : F \to X$ with $\|S\| < 1 + \varepsilon$ and $Sx = x$ for $x \in X \cap F$. For the case when $Y = X^{**}$ this is known as the Principle of Local Reflexivity (see [48]); see also [43] for similar results in the isomorphic version.

In this paper we will be concerned with special classes of ideals where additional constraints are imposed on the projections. There is an extensive

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literature concerning a special class of ideals known as M-ideals ([1], [34]). We say that X is an *M-ideal* in Y if $X^\perp$ is an L-summand in Y^*, i.e. there is a projection $P : Y^* \to Y^*$ with $\ker P = X^\perp$ and $\|y^*\| = \|Py^*\| + \|y^* - Py^*\|$ for all $y^* \in Y^*$. Thus in this case $Y^* = V \oplus_1 X^\perp$ where $V = P(Y^*)$. We say X is an M-ideal (without reference to any other space) if it is an M-ideal in X^{**}.

In [10] a somewhat less restrictive notion was introduced. We say that X is a *u-ideal* in Y if there is a projection $P : Y^* \to Y^*$ with $\ker P = X^\perp$ satisfying $\|I - 2P\| = 1$. Equivalently, if $\chi \in X^\perp$ and $\phi \in V$ then $\|\phi + \chi\| = \|\phi - \chi\|$. In [10] it is shown that if X is a separable reflexive space with the approximation property then the space of compact operators $\mathcal{K}(X)$ is a u-ideal in $\mathcal{L}(X)$ (its bidual) if and only if X has the so-called unconditional metric approximation property (UMAP).

For complex Banach spaces, there is a natural strengthening of the definition. We say that a complex Banach space X is an *h-ideal* in Y if there is an hermitian projection $P : Y^* \to Y^*$ with $\ker P = X^\perp$. This is equivalent to requiring that if $\chi \in X^\perp$, $\phi \in V$ and $|\lambda| = 1$ then $\|\phi + \chi\| = \|\phi + \lambda\chi\|$.

The natural examples of u-ideals or h-ideals (with respect to their biduals) are order-continuous Banach lattices. Much of this paper is devoted to a general study of u-ideals and h-ideals and it will be seen that they inherit many of the properties of order-continuous Banach lattices. It should be stressed, however, that there are many examples of u-ideals (or h-ideals) which are not Banach lattices (for example any M-ideal and, as we shall see in Section 8, certain spaces of operators). It will also be seen that the theory of h-ideals is considerably more satisfactory than the theory of u-ideals because of our ability to exploit the rich theory of hermitian operators.

We now turn to a detailed discussion of our results. In Section 2, motivated by some ideas of Godun [33] we introduce the Godun set of a Banach space X, $G(X)$, which is defined to be the set of all scalars λ such that $\|I - \lambda\pi\| = 1$ where π is the canonical projection of X^{***} onto X^*. If X contains a copy of ℓ_1 then the Godun set $G(X)$ reduces to $\{0\}$. If X is separable and X^* is nonseparable then $G(X) \subset [0,1]$ and equality is possible with $X = JT$. If X^* is separable and if $1 < \lambda < 2$ we show that X can be renormed so that $[0,\lambda] \subset G(X)$. This gives us an improvement of a result of Finet–Schachermayer [18] on renormings so that the characteristic of every proper closed subspace of X^* is at most $1/2 + \varepsilon$; we show that such a renorming can be made so that the same property is inherited by subspaces and quotients.

In Section 3 we make a preliminary study of u-ideals in a given space Y and show for example that if X is a u-ideal in Y which contains no copy of c_0 then X is a u-summand (i.e. there is a projection Q onto X with $\|I - 2Q\| = 1$).

In Section 4 we make some preliminary observations about u-ideals and h-ideals (with respect to their biduals), and give some examples. If X is a u-ideal such that the range V of the induced projection on X^{***} is norming then X is called a strict u-ideal. If X is separable or does not contain ℓ_1, it then follows (Section 5) that $V = X^*$. M-ideals are automatically strict u-ideals, but in Section 4 we give an example to show that there exist strict u-ideals which cannot be renormed to be M-ideals. In [27] (for a special case see [30]) it was shown that M-ideals have Pełczyński's property (u) with constant one. In Section 5 we show that a separable Banach space not containing ℓ_1 is a strict u-ideal if and only if it has property (u) with constant one. Our argument is rather different and more elementary than in [27]. There is an analogous characterization of strict h-ideals in terms of the complex version of property (u).

In Section 6, we turn to the study of general h-ideals (with no strictness assumptions). Here, using the powerful theory of hermitian operators [4], [5], we give a complete characterization of separable h-ideals in Theorem 6.5. Let X be a separable complex Banach space and let $Ba(X)$ denote the subspace of all $x^{**} \in X^{**}$ which are of the first Baire class on B_{X^*} for the weak*-topology, or equivalently $x^{**} \in Ba(X)$ if and only if there is a sequence (x_n) in X converging weak* to x^{**}. We show that if X is an h-ideal there is an hermitian projection of X^{**} onto $Ba(X)$ and X has the complex property (u) with constant one. These two conditions, however, are not sufficient to make X an h-ideal; the example is presented later in Section 8. The necessary and sufficient condition is that there is an hermitian projection T onto $Ba(X)$ so that for every $x^{**} \in X^{**}$ there is a sequence (x_n) in X such that $\lim_{n\to\infty} x_n = Tx^{**}$ weak* and $\limsup_{n\to\infty} \|x^{**} - (1+\lambda)x_n\| = \|x^{**}\|$ whenever $|\lambda| = 1$.

We conclude Section 6 by studying subspaces of h-ideals and applying our results to subspaces of order-continuous lattices. Thus for example a subspace of L_1 is an h-ideal if and only if it is nicely placed ([25]).

Section 7 considers the less satisfactory theory of u-ideals. Our results here are not nearly as complete as in the complex (h-ideal) case. A typical example is that if X is an h-ideal containing no copy of ℓ_1 then X is necessarily a strict h-ideal, but the corresponding result for u-ideals is known (Theorem 7.4) only under certain restrictive hypotheses (e.g. that X has property (u) with constant less than two). We also give a partial result on the problem of whether c_0 is isomorphically the only predual of ℓ_1 with property (u).

In Sections 8 and 9 we consider operator spaces. In Section 8, which may be read directly after Section 6, we consider two basic problems: characterize those X so that $\mathcal{K}(X)$ is a u-ideal or an h-ideal in $\mathcal{L}(X)$ and characterize those X so that $\mathcal{K}(X)$ is an h-ideal (in $\mathcal{K}(X)^{**}$). In fact, if we assume X

is separable, reflexive and has the approximation property, both problems were considered (in the u-ideal case) in [10] where it is shown that $\mathcal{K}(X)$ is a u-ideal in $\mathcal{L}(X)$ if and only if X has the unconditional metric approximation property (UMAP). We extend this result in Theorem 8.3, in particular omitting the hypothesis of the approximation property, and considering certain nonreflexive spaces.

The problem of when $\mathcal{K}(X)$ can be an h-ideal in $\mathcal{K}(X)^{**}$ turns out to have a rather intriguing answer. If we assume that X is separable and has the metric compact approximation property it is not difficult to see that two necessary conditions are that X^* is separable and that X has the complex (UKAP), i.e. given $\varepsilon > 0$ there is a sequence (A_n) of compact operators such that $x = \sum_{n=1}^{\infty} A_n x$ for $x \in X$ and for every $n \in \mathbb{N}$, and $|\alpha_k| \le 1$ for $1 \le k \le n$, $\|\sum_{k=1}^{n} \alpha_k A_k\| < 1+\varepsilon$. Under these two conditions $Y = \mathcal{K}(X)$ has complex property (u) with constant one and there is an hermitian projection of Y^{**} onto $Ba(Y)$. Nevertheless, these conditions are not sufficient for $\mathcal{K}(X)$ to be an h-ideal. In Theorem 8.6 we show that if X is separable and has the metric compact approximation property then $\mathcal{K}(X)$ is an h-ideal if and only if X is an M-ideal with complex (UKAP). It is then easy to give examples of spaces X with a 1-unconditional basis so that X^* is separable and yet X is not an M-ideal, and for such spaces $\mathcal{K}(X)$ provides the example promised in Section 6.

In Section 9, we relate u-ideals to a problem suggested by [10]: if a separable Banach space has (UMAP), does it have commuting (UMAP)? Finally, in Section 10 we take the opportunity to list some problems which arise in connection with this work.

Before continuing, we introduce some notation. If X is a Banach space, real or complex, we denote $B_X = \{x : \|x\| \le 1\}$ and $S_X = \{x : \|x\| = 1\}$. We use $\mathbb{K}$ to denote either $\mathbb{R}$ or $\mathbb{C}$.

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2. The Godun set and renormings of separable Banach spaces. We begin by discussing a general situation related to the notion of locally complemented subspaces introduced in [43]. A subspace X of a Banach space Y is *locally complemented* if there is a constant λ such that whenever F is a finite-dimensional subspace of Y then there is a linear operator $T : F \to X$ with $Tx = x$ for $x \in F \cap X$ and $\|T\| \le \lambda$. It is shown in [43, Theorem 3.5] that this is equivalent to the existence of a bounded projection P on Y^* whose kernel is $X^\perp$ (see condition (4) of [43, Theorem 3.5]). Our first results will relate to this general situation, although we will be specifically applying the

results to the case $Y = X^{**}$ with P being the canonical projection of X^{***} onto X^*. (In this case the local complementation of X in X^{**} is essentially the Principle of Local Reflexivity [48].)

Let Y be a (real or complex) Banach space and suppose that X is a closed subspace of Y. Let us suppose that P is a bounded projection on Y^* such that $\ker P = X^\perp$; we set $V = P(Y^*)$. Let $R_X : Y^* \to X^*$ be the natural restriction map. Then we may define a bounded operator $T : Y \to X^{**}$ such that for $y \in Y$, $y^* \in Y^*$ we have

$$\langle R_X y^*, Ty\rangle = \langle y, Py^*\rangle .$$

It is clear that $\|T\| \le \|P\|$ and that if $x \in X$ then $Tx = x$. Let us also note that we will preserve the notation introduced here for P, T, V throughout the paper. However, when dealing with spaces of operators we will use $\mathcal{P}, \mathcal{T}$ in place of P, T.

LEMMA 2.1. *The following conditions are equivalent*:

(1) V *is weak***-closed.*
(2) P *is weak***-continuous.*
(3) $T(Y) \subset X$.

Proof. (1)$\Leftrightarrow$(2) is standard. For (2)$\Rightarrow$(3) let $P = Q^*$ where $Q : Y \to Y$. It is readily seen that Q is a projection of Y onto X and hence that $Q = T$. For (3)$\Rightarrow$(2) simply observe that $P = T^*$. ∎

LEMMA 2.2. *Suppose K is a compact subset of $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and that $a > 0$. Then the following conditions are equivalent*:

(1) $\|I - \lambda P\| \le a$, *whenever* $\lambda \in K$.

(2) *Whenever $\varepsilon > 0$, $y \in Y$ and A is a convex subset of X such that Ty is in the weak***-closure of A then there exists $x \in A$ with $\|y - \lambda x\| < a\|y\| + \varepsilon$, for all $\lambda \in K$.*

(3) *For any $y \in Y$, there is a net (x_d) in X such that $\lim_d x_d = Ty$ for the weak***-topology and $\limsup_d \|y - \lambda x_d\| \le a\|y\|$, for all $\lambda \in K$.*

Proof. (1)$\Rightarrow$(2). For convenience we suppose that $\|y\| = 1$. It clearly suffices to consider the case when K is finite, say $K = \{\lambda_k\}_{k=1}^{n}$. If the conclusion is false then by the Hahn–Banach theorem there exist $y_k^* \in Y^*$ with $\sum_{k=1}^{n} \|y_k^*\| \le 1$ and such that $\sum_{k=1}^{n} \Re y_k^*(y - \lambda_k x) > a + \frac{1}{2}\varepsilon$ for all $x \in A$. Thus

$$\sum_{k=1}^{n} \Re\langle y, (I - \lambda_k P)y_k^*\rangle + \sum_{k=1}^{n} \Re\lambda_k\langle y, Py_k^*\rangle - \sum_{k=1}^{n} \Re\lambda_k\langle x, Py_k^*\rangle \ge a + \frac{\varepsilon}{2}$$

for $x \in A$. This can be rewritten as

$$\sum_{k=1}^{n} \Re\langle y, (I - \lambda P)y_k^*\rangle + \sum_{k=1}^{n} \Re(\lambda_k\langle R_X y_k^*, Ty - x\rangle) \ge a + \frac{\varepsilon}{2} .$$

Now since Ty is in the weak*-closure of A this implies that

$$\sum_{k=1}^{n}|\langle y,(I-\lambda_k P)y_k^*\rangle| \geq a+\frac{\varepsilon}{2} > a\,,$$

which is a contradiction.

(2)⇒(3). It is enough to take any convex weak*-neighborhood W of Ty and apply (2) to $A=W\cap X$.

(3)⇒(1). Let $\lambda\in K$. Suppose $y^*\in Y^*$ with $\|y^*\|=1$ and suppose $y\in Y$ with $\|y\|=1$. Then we may find a net (x_d) in X with $\lim x_d = Ty$ weak* and $\limsup\|y-\lambda x_d\|\leq a$. Then

$$\begin{aligned}\langle y,(I-\lambda P)y^*\rangle &= \langle y,y^*\rangle - \lambda\langle R_X y^*, Ty\rangle\\ &= \langle y,y^*\rangle - \lambda\lim_d\langle R_X y^*, x_d\rangle = \lim_d\langle y-\lambda x_d, y^*\rangle\end{aligned}$$

and the lemma follows. ■

Let us now consider a special case when $Y=X^{**}$ and $P=\pi$ is the canonical projection of X^{***} onto X^* (or more precisely $j_1(X^*)$) given formally by $\pi = j_1 j_0^*$, where $j_0 : X\to X^{**}$ and $j_1 : X^*\to X^{***}$ are the canonical embeddings; of course $\ker\pi = X^\perp$. In this case T reduces to the identity operator on X^{**}. We then have:

PROPOSITION 2.3. *The following conditions are equivalent*:

(1) $\|I-\lambda\pi\|\leq a$.

(2) *For any $\varepsilon>0$, and for any $x^{**}\in X^{**}$ and any convex subset A of X such that x^{**} is in the weak*-closure of A there exists $x\in A$ with $\|x^{**}-\lambda x\| < a\|x^{**}\|+\varepsilon$.*

(3) *For any $x^{**}\in X^{**}$ there is a net (x_d) in X such that $\lim_d x_d = x^{**}$ weak* and $\limsup_d\|x^{**}-\lambda x_d\|\leq a\|x^{**}\|$.*

(4) *For any sequence (x_n) in B_X and any $\varepsilon>0$ there exist n and $u\in \operatorname{co}\{x_k\}_{k=1}^n$, $t\in\operatorname{co}\{x_k\}_{k=n+1}^\infty$ with $\|t-\lambda u\|<a+\varepsilon$.*

Proof. The equivalence of the first three conditions is immediate from Lemma 2.2. Let us prove (2)⇒(4). Let x^{**} be a weak*-cluster point of the sequence (x_n). Letting A be the convex hull of $\{x_n\}_{n=1}^\infty$ and applying (2) we find $u\in\operatorname{co}\{x_k\}_{k=1}^n$ for some n such that $\|x^{**}-\lambda u\|<a\|x^{**}\|+\frac13\varepsilon$. Now if $\|t-\lambda u\|\geq a+\varepsilon$ for all $t\in\operatorname{co}\{x_k\}_{k>n}$ it follows by the Hahn–Banach theorem that there exists $x^*\in B_{X^*}$ such that $x^*(x_k-\lambda u)\geq a+\frac23\varepsilon$ for all $k\geq n+1$, which leads to a contradiction and so (4) is proved.

Now let us show that (4) implies (3). Indeed, if (3) fails we can find $x^{**}\in S_{X^{**}}$, $\varepsilon>0$ and a closed convex weak*-neighborhood W of x^{**} with $\|x^{**}-\lambda w\|>a+\varepsilon$ for all $w\in W\cap X$. Now by induction we may pick a sequence $(x_n)_{n\geq1}$ in $W\cap X$ and a decreasing sequence $(W_n)_{n\geq1}$ of weak*-closed convex neighborhoods of x^{**} so that $x_n\in W_{n-1}$ and if $w\in W_n$ and $x\in C_n=\operatorname{co}\{x_k\}_{k=1}^n$ then $\|w-\lambda x\|>a+\varepsilon$. To start the induction pick $x_1\in W$ and then let $W_1\subset W$ be chosen to be a weak*-closed convex neighborhood of x^{**} such that $\|w-\lambda x_1\|>a+\varepsilon$ on W_1. Next assume that $x_1,\dots,x_{n-1}$, $W_1,\dots,W_{n-1}$ have been chosen where $n\geq2$. Since $\lambda C_{n-1}+(a+\varepsilon)B_{X^{**}}$ is weak*-compact and fails to include x^{**} we can find a closed convex weak*-neighborhood $W_n\subset W_{n-1}$ of x^{**} so that if $w\in W_n$ and $x\in C_{n-1}$ then $\|w-\lambda x\|>a+\varepsilon$. Then pick $x_n\in W_n$. This completes the induction and now if $u\in C_n$ while $t\in\operatorname{co}\{x_k\}_{k>n}$ then $t\in W_n$ and hence $\|t-\lambda u\|>a+\varepsilon$, which contradicts (4). ■

Let us now define the *Godun set* $G(X)$ of X (cf. [32], [33]) to be the collection of all scalars λ such that $\|I-\lambda\pi\|=1$. It is routine to observe that $0\in G(X)$, and that $G(X)$ is a closed convex set. Further notice that we have $(I-\lambda\pi)(I-\mu\pi)=I-(\lambda+\mu-\lambda\mu)\pi$ so that $\lambda,\mu\in G(X)$ implies $1-(1-\lambda)(1-\mu)\in G(X)$.

LEMMA 2.4. (1) *In the real case $G(X)\subset[0,2]$ and in the complex case $G(X)\subset\{\lambda:|\lambda-1|\leq1\}$.*

(2) *If $G(X)$ contains any $\lambda\neq0$ then $G(X)$ contains the interval $[0,1]$.*

(3) *If $G(X)$ contains any $\lambda\notin[0,1]$ then $G(X)$ contains some $\lambda>1$.*

Proof. Notice that $1-\alpha\in G(X)$ implies $1-\alpha^n\in G(X)$, for every n. This gives (1). For (2) notice that if $1-\alpha\in G(X)$ for some $\alpha\neq1$ then we can find such a point with $|\alpha|<1$ by a convexity argument and then iteration shows that 1 is in $G(X)$. (3) is obvious in the real case; for the complex case suppose $1-re^{i\theta}\in G(X)$ where $r>0$, and θ is not a multiple of π. If $G(X)$ does not contain $1+\delta$ for any $\delta>0$ then there exist, by a separation argument, $a,b\in\mathbb{R}$ such that $a^2+b^2=1$, $a\geq0$ and $r^n(a\cos n\theta+b\sin n\theta)\geq0$ for all n. This is clearly impossible. ■

LEMMA 2.5. *Let Z be a closed subspace of X. Then $G(Z)\supset G(X)$ and $G(X/Z)\supset G(X)$. Furthermore, $\lambda\in G(X)$ if and only if $\lambda\in G(Z)$ for every separable subspace Z of X.*

Proof. These observations follow very quickly from equivalence (4) of Proposition 2.3. ■

The next proposition extends Proposition 2 of [8].

PROPOSITION 2.6. *If X contains a subspace isomorphic to ℓ_1 then for every $\lambda\in\mathbb{K}$ we have $\|I-\lambda\pi\|=1+|\lambda|$ and so $G(X)=\{0\}$.*

Proof. For every $\varepsilon>0$, X contains a sequence (x_n) in the unit ball such that for all $\alpha_1,\dots,\alpha_n$ we have $\|\sum_{k=1}^n\alpha_n x_n\|\geq(1-\varepsilon)\sum_{k=1}^n|\alpha_k|$ (see [40]). The result follows easily from Proposition 2.3(4). ■

Let us recall that if X is a Banach space and M is a closed subspace of X^* then the *characteristic* $r(M)$ of M is defined to be the greatest constant

r such that

$$\sup_{\substack{x^* \in M \\ \|x^*\| \le 1}} |x^*(x)| \ge r\|x\|$$

for every $x \in X$.

PROPOSITION 2.7. *Suppose $1 < \lambda \le 2$ and that $\|I - \lambda\pi\| = a < \lambda$. Then for any proper closed subspace M of X^* we have $r(M) \le a\lambda^{-1}$.*

Proof. It is enough to consider the case when $M = \ker x^{**}$ where $x^{**} \in X^{**}$ with $\|x^{**}\| = 1$. Let (x_d) be a net in B_X with $\lim x_d = x^{**}$ weak* and $\limsup \|x^{**} - \lambda x_d\| \le a$. Then

$$\lambda \sup_{\substack{x^* \in M \\ \|x^*\| \le 1}} |x^*(x_d)| \le \|x^{**} - \lambda x_d\|$$

for all d and the proposition follows. ∎

The following is now immediate.

PROPOSITION 2.8. *Let X be a separable Banach space so that $G(X)$ contains some $\lambda > 1$, or $\|I - 2\pi\| < 2$. Then X^* is separable.*

We now turn to a converse statement.

THEOREM 2.9. *Let X be a separable Banach space for which X^* is separable. If $1 < \lambda < 2$ then X can be equivalently normed so that $\lambda \in G(X)$.*

Proof. We first note a recent deep result of Zippin [64] that X is isomorphic to a closed subspace of a Banach space with a shrinking basis. It therefore suffices to consider the case when X has a shrinking basis. Let (S_n) denote the associated partial sum operators. Then ([10]) X can be equivalently normed so that $\|S_n\| = \|I - \lambda S_n\| = 1$ for all n. Now suppose $x^{**} \in X^{**}$ with $\|x^{**}\| = 1$. Then $x_n = S_n^{**}x^{**}$ converges weak* to x^{**} and $\|x^{**} - \lambda x_n\| \le 1$. The result now follows. ∎

COROLLARY 2.10. *If X is a separable Banach space with separable dual and if $\varepsilon > 0$ then X can be equivalently renormed so that any subspace Z of a quotient space of X has the property that whenever M is a proper closed subspace of Z^* then $r(M) \le 1/2 + \varepsilon$.*

Remark. This improves the main result of [18]. There are several consequences which follow for such a space. It follows that for any $\varepsilon > 0$ and any Banach space X with a separable dual there is a renorming so that every basic sequence with basis constant less than $2 - \varepsilon$ is shrinking ([17, Corollary 1.4]). More generally ([31, Proposition 4.3]), any subspace of X, under this renorming, with the $(2 - \varepsilon)$-commuting bounded approximation property in fact has the metric approximation property (or the commuting metric approximation property [10]). Also any subspace or quotient of X with distance less than $2 - \varepsilon$ to a dual space is automatically reflexive (this improves some results from [14]).

Let us conclude by mentioning that the James tree space JT provides an example where $\|I - \pi\| = 1$ but JT^* is nonseparable (see [7]). Of course by Proposition 2.8 there is no renorming of JT such that $\|I - \lambda\pi\| = 1$ for any $\lambda > 1$.

3. u-ideals and h-ideals. We now introduce some terminology. Suppose Y is a real or complex Banach space. A closed subspace X of Y is a *summand* if there is a contractive projection of Y onto X. We will say that a closed subspace X of Y is a *u-summand* if there is subspace Z (the *u-complement* of X) so that $X \oplus Z = Y$ and if $x \in X$, $z \in Z$ then $\|x + z\| = \|x - z\|$. If Y is a complex Banach space we will say that X is an *h-summand* with *h-complement* Z if $X \oplus Z = Y$ and if $x \in X$, $z \in Z$ and $|\lambda| = 1$ then $\|x + \lambda z\| = \|x + z\|$. If X is a u-summand then the induced projection $P : Y \to X$ with $P(Y) = X$ and $\ker P = Z$ satisfies $\|I - 2P\| = 1$. If X is an h-summand then $\|I - (1+\alpha)P\| = 1$ whenever $|\alpha| = 1$ and this is equivalent to the requirement that P is hermitian (cf. [4], [5]).

By way of motivation, let us also mention that X is called an *L-summand* (resp. *M-summand*) in Y if there is a subspace Z so that $X \oplus Z = Y$ and $\|x+z\| = \|x\| + \|z\|$ (resp. $\|x+z\| = \max(\|x\|, \|z\|)$ for every $x \in X$, $z \in Z$. See [34] for more details.

The most elementary example of a u-summand is any band in an order-continuous (or, more generally, order-complete) Banach lattice; in the complex case this will yield an example of an h-summand. In particular, any subsequence of a 1-unconditional basis generates a u-summand (or an h-summand in the complex case). Let us mention two slightly more subtle examples. Set X to be the subspace of $C[0,1]$ of all f such that $f(1-t) = f(t)$. Then X is a u-summand which is not an M-summand; in the case of complex scalars X is a u-summand which fails to be an h-summand. Similarly, if X is the subspace of $L_1[0,1]$ of all f such that $f(x) = f(1-x)$ then X is a u-summand which fails to be an L-summand. These examples demonstrate that the concept of a u-summand is much broader than the existing notions of L-summands and M-summands.

We first observe that if X is a u-summand (and *a fortiori* if X is an h-summand) then its u-complement Z is unique, and is also a u-summand (or an h-summand if X is an h-summand).

LEMMA 3.1. *Suppose X is a closed subspace of Y. Then there is at most one projection P of Y onto X satisfying $\|I - 2P\| = 1$.*

Proof. Suppose Q is another projection onto X with $\|I - 2Q\| = 1$. Then $(I - 2P)(I - 2Q) = I + 2(Q - P)$ since $QP = P$, $PQ = Q$. Thus $((I - 2P)(I - 2Q))^n = I + 2n(Q - P)$ and so $P = Q$. ∎

We recall from the introduction that a closed subspace X of Y is an *ideal* in Y if $X^\perp$ is the kernel of a contractive projection on Y^*. (This is equivalent to the existence of a linear norm-preserving extension operator from X^* to Y^*; see [38], [58] where it is essentially shown that every separable subspace of a nonseparable Banach space is contained in a separable ideal.) Next we will say that X is a *u-ideal* in Y if $X^\perp$ is a u-summand in Y^*, and an *h-ideal* if $X^\perp$ is an h-summand. We recall that X is an M-ideal in Y if $X^\perp$ is an L-summand in Y^*. Thus a real M-ideal is a u-ideal and a complex M-ideal is an h-ideal.

The simplest example of a u-ideal is an order-ideal in an arbitrary Banach lattice; indeed, if Y is a Banach lattice and X is an order-ideal then $X^\perp$ is a band in Y^*. Again, in the complex case, this will give an example of an h-ideal. Later, in Section 4, we will discuss examples of spaces X which are u-ideals in their biduals and give other examples.

Assuming that X is a u-ideal in Y, let V be the u-complement of $X^\perp$ in Y^*; let P be a projection on Y^* so that $\|I-2P\|=1$ and whose range is V and $\ker P = X^\perp$.

Lemma 2.1 yields immediately:

LEMMA 3.2. *If X is a u-ideal in Y then X is a u-summand if and only if V is weak*-closed.*

Proof. Obviously if V is weak*-closed then P is weak*-continuous and so $P=Q^*$ where $\|I-2Q\|=1$ and $Q(Y)=X$. Conversely, suppose X is a u-summand and let Q be a projection onto X with $\|I-2Q\|=1$. Then $I-Q^*$ has range $X^\perp$ and so $I-Q^*=I-P$ by Lemma 3.1. Hence P is weak*-continuous. ■

Motivated by this lemma we will say that X is a *strict u-ideal or h-ideal* if V is a norming subspace of Y^*.

If X is an arbitrary Banach space and $x^{**}\in X^{**}$ we will define its *u-constant* $\kappa_{\rm u}(x^{**})$ to be the infimum of all a such that we can write $x^{**}=\sum_{n=1}^\infty x_n$ in the weak*-topology with $x_n\in X$ and such that for any n and $\theta_k=\pm1$ for $1\le k\le n$ we have

$$\Big\|\sum_{k=1}^n \theta_k x_k\Big\|\le a\,.$$

We set $\kappa_{\rm u}(x^{**})=\infty$ if no such a exists. Let $Ba(X)$ be the collection of $x^{**}\in X$ such that there is a sequence (x_n) in X with $\lim x_n = x^{**}$. We recall that X has property (u) if every $x^{**}\in Ba(X)$ has $\kappa_{\rm u}(x^{**})<\infty$. It then follows from the closed graph theorem that there is a constant C so that $\kappa_{\rm u}(x^{**})\le C\|x^{**}\|$ for all $x^{**}\in Ba(X)$. The least such constant C will be denoted by $\kappa_{\rm u}(X)$.

If X is a complex Banach space we define also $\kappa_{\rm h}(x^{**})$ to be the infimum of all a so that $x^{**}=\sum_{n=1}^\infty x_n$ and for any n, and any $|\theta_k|=1$ for $1\le k\le n$, we have

$$\Big\|\sum_{k=1}^n \theta_k x_k\Big\|\le a\,.$$

Clearly $\kappa_{\rm u}(x^{**})\le\kappa_{\rm h}(x^{**})\le 2\kappa_{\rm u}(x^{**})$. If X has property (u) we define $\kappa_{\rm h}(X)$ to be the least constant C so that $\kappa_{\rm h}(x^{**})\le C\|x^{**}\|$ for $x^{**}\in Ba(X)$. Thus $\kappa_{\rm u}(X)\le\kappa_{\rm h}(X)\le 2\kappa_{\rm u}(X)$.

We now prove a lemma which refines Lemma 2.2. Assume X is a u-ideal in Y, and that P is the associated projection on Y^* with $\ker P=X^\perp$. We use the definition of the operator $T:Y\to X^{**}$ from the previous section so that for $y\in Y$, $y^*\in Y^*$, $\langle R_X y^*, Ty\rangle=\langle y, Py^*\rangle$. In this case $\|T\|\le\|P\|=1$ and of course $Tx=x$ if $x\in X$.

LEMMA 3.3. *Suppose that X is a u-ideal (resp. an h-ideal) in Y. Suppose that $y\in Y$, and $\varepsilon>0$. Let A be a convex subset of X such that Ty is in the weak*-closure of A and that B is any compact subset of X. Then there exists $x\in A$ such that $\|y-(1+\lambda)x-\lambda z\|<\|y+z\|+\varepsilon$ whenever $-1\le\lambda\le1$ (resp. $|\lambda|\le1$) and $z\in B$.*

Proof. We give the proof in the h-ideal case as this is slightly more complicated. We may assume that $0\in B$. Let $M=\max\{\|z\|:z\in B\}$ and pick $0<\delta<1$ so that $(M+4+2\|y\|)\delta<\varepsilon$.

Let $\{\lambda_1,\dots,\lambda_m\}$ be a δ-net for the closed unit disk, which we suppose includes zero, and let $\{z_1,\dots,z_n\}$ be a δ-net for B, also including zero. For any subset $\mathcal J$ of $\Omega=[m]\times[n]$ define $H_{\mathcal J}$ to be the set of $x\in A$ such that

$$\|y-(1+\lambda_j)x-\lambda_j z_k\|<\|y+z_k\|+\delta$$

whenever $(j,k)\in\mathcal J$. ($H_\emptyset=A$.) We will show that H_Ω is nonempty. In fact, if H_Ω is empty then there is a maximal proper subset $\mathcal J$ of Ω such that Ty is in the weak*-closure of $H_{\mathcal J}$. Pick any $(j,k)\notin\mathcal J$. Then there is a weak*-open convex set W in X^{**} containing Ty such that $W\cap H_{\mathcal K}=\emptyset$ where $\mathcal K=\mathcal J\cup\{(j,k)\}$. However, $A'=W\cap H_{\mathcal J}$ is convex and Ty is in its weak*-closure. Thus $T(y+z_k)$ is in the weak*-closure of $A'+z_k$. Now by Lemma 2.2 there exists $x\in W\cap H_{\mathcal J}$ such that $\|y+z_k-(1+\lambda_j)(x+z_k)\|<\|y+z_k\|+\delta$. On re-organization this implies that $x\in W\cap H_{\mathcal K}$ contrary to assumption. It follows that H_Ω is nonempty. Pick any $x\in H_\Omega$. Then if $z\in B$ and $|\lambda|\le1$ we may find $(j,k)\in\Omega$ such that $|\lambda-\lambda_j|\le\delta$ and $\|z-z_k\|\le\delta$. Thus

$$\begin{aligned}\|y-(1+\lambda)x-\lambda z\|&\le\|y-(1+\lambda_j)x-\lambda_j z_k\|+\delta(1+\|x\|+\|z_k\|)\\&\le\|y+z_k\|+\delta(2+\|x\|+\|z_k\|)\\&\le\|y+z\|+\delta(3+\|x\|+\|z_k\|)\,.\end{aligned}$$

Zero appears in both δ-nets so we have $\|y-x\| \le \|y\|+\delta$ and thus $\|x\| \le 2\|y\|+1$. Hence $\delta(3+\|x\|+\|z_k\|) \le (M+4+2\|y\|)\delta < \varepsilon$, and the lemma is established. ∎

The next lemma is fundamental for the results of the paper.

LEMMA 3.4. *Assume X is a u-ideal (resp. an h-ideal) in Y. Suppose $y \in Y$, $\varepsilon > 0$, and that (A_n) is a sequence of convex subsets of X such that Ty is in the weak*-closure H_n of each A_n. Then there exists $x^{**} \in \bigcap_n H_n$ with $\kappa_u(x^{**}) \le \|y\|+\varepsilon$ (resp. $\kappa_h(x^{**}) \le \|y\|+\varepsilon$) and such that $d(x^{**},X) \ge \frac{1}{2}d(Ty,X)$. In particular, if $Ty \in Ba(X)$ then $\kappa_u(Ty) \le \|y\|$ (resp. $\kappa_h(Ty) \le \|y\|$).*

Proof. Let us assume that X is a u-ideal (resp. an h-ideal) in Y and that $\|y\|=1$. Let $s_0=0$. Then we will construct by induction a sequence $(s_n)_{n=1}^{\infty}$ in X and a sequence of weak*-open convex neighborhoods $(W_n)_{n=1}^{\infty}$ of Ty so that if $x_n = s_n - s_{n-1}$ and F_n is the linear span of $\{s_0,\dots,s_{n-1}\}$ for $n \ge 1$, then we have $d(w,F_n) > (1-1/n)d(Ty,X)$ for $w \in W_n$, $s_n \in W_n \cap A_n$, and

$$\Big\|y - s_n - \sum_{j=1}^{n}\theta_j x_j\Big\| < 1+\varepsilon$$

whenever $|\theta_j| \le 1$ for $1 \le j \le n$. Suppose the construction has been carried out to determine $\{s_k\}_{k=0}^{m-1}$ and $\{W_k\}_{k=1}^{m-1}$. We will show how to determine s_m and W_m. In fact, it is clear that the seminorm $x^{**} \to d(x^{**},F_m)$ is weak*-continuous on X^{**} and so we may clearly determine a convex open neighborhood W_m of Ty with the specified property. Now let $0 < \alpha < \varepsilon$ be such that

$$\Big\|y - s_{m-1} - \sum_{j=1}^{m-1}\theta_j x_j\Big\| < 1+\alpha$$

whenever $|\theta_j| \le 1$. Let B be the collection of all $z = s_{m-1} + \sum_{k=1}^{m-1}\theta_j x_j$ for $|\theta_j| \le 1$. According to Lemma 3.3 it is possible to find $s_m \in A_m \cap W_m$ so that whenever $|\lambda| \le 1$ and $z \in B$,

$$\|y-(1+\lambda)s_m - \lambda z\| < \|y+z\| + \varepsilon - \alpha\,.$$

Thus if $|\theta_j| = 1$ for $1 \le j \le m$,

$$\Big\|y - s_m - \sum_{j=1}^{m}\theta_j x_j\Big\| = \Big\|y-(1+\theta_m)s_m - \theta_m\Big(s_{m-1} + \sum_{j=1}^{m-1}\theta_m^{-1}\theta_j x_j\Big)\Big\| < 1+\varepsilon\,.$$

By a convexity argument we obtain the inductive hypothesis.

Now the series $\sum x_j$ is a w.u.c. series which converges to some $x^{**} \in X^{**}$ such that $\kappa_u(x^{**}) \le 1+\varepsilon$ (resp. $\kappa_h(x^{**}) \le 1+\varepsilon$). Clearly $x^{**} \in \bigcap H_n$. Further suppose $x \in X$, and let $\gamma = \|x^{**}-x\|$. Then for any m and any $\delta > 0$, there exist $n \ge m$ and $t \in \mathrm{co}\{s_{m+1},\dots,s_n\}$ such that $\|t-x\| \le \gamma+\delta$. Similarly, there exist $p > n$ and $u \in \mathrm{co}\{s_{n+1},\dots,s_p\}$ such that $\|u-x\| \le \gamma+\delta$. Hence $\|t-u\| \le 2(\gamma+\delta)$. However, $u \in W_n$ so that $\|t-u\| \ge (1-1/n)d(Ty,X)$. It follows that $d(Ty,X) \le 2\gamma$ and so $d(x^{**},X) \ge \frac{1}{2}d(Ty,X)$. ∎

THEOREM 3.5. *Let Y be a Banach space and let X be a u-ideal in Y. Suppose that X contains no copy of c_0. Then X is a u-summand in Y.*

Proof. We need only show that $T(Y) \subset X$ and then apply Lemma 2.1. Now if $y \in Y$ with $\|y\|=1$ we let $A_n = B_X$ for all n and apply Lemma 3.4. We conclude that there exists $x^{**} \in X^{**}$ with $d(x^{**},X) \ge \frac{1}{2}d(Ty,X)$ and $\kappa_u(x^{**}) < \infty$. Thus ([3]) $x^{**} \in X$ and so $Ty \in X$. ∎

Before proceeding further we give a local formulation of the notion of a u-ideal or an h-ideal.

PROPOSITION 3.6. *Let Y be a Banach space and let X be a closed subspace of Y. In order that X be a u-ideal (resp. an h-ideal) in Y it is necessary and sufficient that for every finite-dimensional subspace F of Y and every $\varepsilon > 0$ there is a linear map $L: F \to X$ such that $Lx = x$ for $x \in F \cap X$ and $\|f-2Lf\| \le (1+\varepsilon)\|f\|$ for every $f \in F$ (resp. $\|f-(1+\lambda)Lf\| \le (1+\varepsilon)\|f\|$ for every $f \in F$, and for every λ such that $|\lambda|=1$).*

Proof. We prove this result only in the u-ideal case; the h-ideal case is a minor modification. Suppose first that X is a u-ideal in Y, and that F is a finite-dimensional subspace of Y. Then we claim that $\mathcal{L}(F,X)$ is a u-ideal in $\mathcal{L}(F,Y)$. In fact, $\mathcal{L}(F,Y)^*$ can be identified with $F \otimes_\pi Y^*$ and so we can induce a projection $\mathcal{P}$ on it by $\mathcal{P}(f \otimes y^*) = f \otimes Py^*$. It is clear that $\|I - 2\mathcal{P}\| = 1$ and that $\ker\mathcal{P} = \mathcal{L}(F,X)^\perp$. Further $\mathcal{P}$ induces a map $\mathcal{T}: \mathcal{L}(F,Y) \to \mathcal{L}(F,X^{**})$ in the usual way, as described at the beginning of Section 2, so that for $\Phi \in \mathcal{L}(F,X)^*$ we have $\mathcal{T}(L)(\Phi) = \mathcal{P}(\Phi)(L)$. Let $J: F \to Y$ be the identity map.

Let $\mathcal{A}$ be the collection of all $L \in \mathcal{L}(F,X)$ such that $Lx = x$ for all $x \in F \cap X$. Suppose $\mathcal{T}(J)$ is not in the weak*-closure of $\mathcal{A}$. Then there exists $\Phi \in \mathcal{L}(F,X)^*$ so that $\sup_{L\in\mathcal{A}} \Re\Phi(L) = \alpha < \Re\mathcal{T}(J)(\Phi)$. Clearly $\Phi(S) = 0$ if $S = 0$ on $F \cap X$. It follows that we can write $\Phi = \sum_{j=1}^{m} f_j \otimes x_j^*$ where $\{f_j\}$ is a basis for $F \cap X$, and $x_j^* \in X^*$. Let y_j^* be extensions of x_j^* to Y^*. Then

$$\langle \Phi, \mathcal{T}(J)\rangle = \Big\langle J, \mathcal{P}\Big(\sum_{j=1}^{m} f_j \otimes y_j^*\Big)\Big\rangle = \sum_{j=1}^{m}\langle J, f_j \otimes Py_j^*\rangle$$

$$= \sum_{j=1}^{m}\langle f_j, Py_j^*\rangle = \sum_{j=1}^{m} x_j^*(f_j)\,.$$

Now letting S be any projection of F onto $F \cap X$ we find that $\Phi(S) = \mathcal{T}(J)(\Phi)$. Thus $\mathcal{T}(J)$ is in the weak*-closure of A and so there exists, by Lemma 3.3, $L \in \mathcal{A}$ so that $\|J - 2L\| < 1 + \varepsilon$.

Conversely, suppose for every finite-dimensional F and $\varepsilon > 0$ there exists $L = L_{F,\varepsilon} : F \to X$ so that $Lx = x$ for $x \in F \cap X$ and $\|f - 2Lf\| \le (1+\varepsilon)\|f\|$ for $f \in F$. We may regard the collection of all pairs (F, ε) as a directed set in the obvious way. Extend $L_{F,\varepsilon}$ to a nonlinear operator $L_{F,\varepsilon} : Y \to X$ by setting $L_{F,\varepsilon}(x) = 0$ for $x \notin F$. By a compactness argument we can find a subnet (L_d) of $L_{F,\varepsilon}$ so that for every $y^* \in Y^*$, $y \in Y$, $\lim y^*(L_d y) = h(y^*, y)$ exists. Then $y \to h(y^*, y)$ is linear and bounded and so we can define $Py^* \in Y^*$ by $\langle y, Py^* \rangle = h(y^*, y)$. Further P is linear, $\ker P = X^\perp$ and $\|I - 2P\| = 1$. ∎

4. Banach spaces X which are u-ideals or h-ideals in X^{**}. In this paper our main interest is in spaces X embedded in their biduals. We will say that X is a u-ideal (resp. an h-ideal) if X is a u-ideal (resp. an h-ideal) in X^{**} for the canonical embedding. By Proposition 3.6, it follows that a space is a u-ideal or an h-ideal if it satisfies a strong form of the Principle of Local Reflexivity.

PROPOSITION 4.1. *A Banach space X is a u-ideal (resp. an h-ideal) if given a finite-dimensional subspace F of X^{**} and $\varepsilon > 0$ there exists a linear map $L : F \to X$ so that $Lx = x$ for $x \in X \cap F$ and $\|f - 2Lf\| \le (1 + \varepsilon)\|f\|$ for $f \in F$ (resp. $\|f - (1 + \lambda)Lf\| \le (1 + \varepsilon)\|f\|$ for $|\lambda| = 1$ and $f \in F$).*

An immediate remark follows.

COROLLARY 4.2. *Suppose X is a Banach space so that for every $\varepsilon > 0$ there is a u-ideal (resp. an h-ideal) Y so that X is $(1 + \varepsilon)$-isomorphic to a $(1 + \varepsilon)$-complemented subspace of Y. Then X is a u-ideal (resp. an h-ideal).*

We will say that X is a *u-summand* (resp. an *h-summand*) if X is a u-summand (resp. an h-summand) in X^{**}. Furthermore we will say that X is a *strict u-ideal* (resp. *strict h-ideal*) if X is a strict u-ideal (resp. strict h-ideal) in X^{**}. In particular (see Proposition 5.2 below for more details), X will be a strict u-ideal if $\|I - 2\pi\| = 1$ so that X^* is a u-complement of $X^\perp$. The following examples illustrate our definitions.

EXAMPLE (1). If X is an order-continuous Banach lattice then (see [56]) X is an order-ideal in X^{**} and so $X^\perp$ is a band in X^{***}. Hence X is a u-ideal (or an h-ideal in the complex case). X is a u-summand if and only if X contains no copy of c_0, i.e. if and only if X is a band in X^{**}. Finally, X is a strict u-ideal if and only if X^{**} is the band generated by X, i.e. if and only if X contains no copy of ℓ_1.

EXAMPLE (2). If X is a real (resp. complex) Banach space which is an M-ideal in X^{**} then X is a strict u-ideal (resp. strict h-ideal).

EXAMPLE (3). If X is a separable reflexive Banach space with the approximation property then ([10, Theorem 3.9]) $\mathcal{K}(X)$ is a u-ideal if and only if X has (UMAP), i.e. if and only if there is a sequence (T_n) of finite-rank operators with $\lim \|I - 2T_n\| = 1$ and $\lim T_n x = x$ for $x \in X$. In fact, the argument of [10] shows that $\mathcal{K}(X)$ is actually a strict u-ideal. In the complex case it is easy to modify this result to yield that $\mathcal{K}(X)$ is a strict h-ideal if and only if X has complex (UMAP), i.e. there is a sequence (T_n) of finite-rank operators so that $\lim \|I - (1 + \lambda)T_n\| = 1$ whenever $|\lambda| = 1$ and so that $\lim T_n x = x$ for all $x \in X$. See Sections 8, 9 for more discussion.

We also mention here that if X is a reflexive space with a 1-unconditional FDD and Y is a closed subspace of X then $\mathcal{K}(Y)$ is a u-ideal in $\mathcal{L}(Y)$ if and only if Y has the compact approximation property (see Theorem 8.3 and the proof of Theorem 9.2(6)⇒(1) below); we refer to [11], [44] and [63] for similar results for M-ideals.

EXAMPLE (4). Suppose X is a reflexive Banach space. It follows from [39] that if X has a 1-symmetric basis and X is different from ℓ_p, $1 < p < \infty$, then $\mathcal{K}(X)$ is a strict u-ideal which is not an M-ideal. It is shown in [44] that if X has a 1-symmetric basis and is not isomorphic to an Orlicz sequence space then X cannot even be renormed so that $\mathcal{K}(X)$ is an M-ideal.

In view of these examples we now wish to show that there exist separable strict u-ideals X which cannot be renormed to be M-ideals. In order to do this we will need a criterion for a separable Banach space to be an M-ideal.

PROPOSITION 4.3. *Let X be a separable Banach space (real or complex). The following conditions on X are equivalent.*

(1) *X is an M-ideal.*

(2) *If $x \in X$ and $x^{**} \in X^{**}$ then there is a net (x_d) in X so that $\lim_d x_d = x^{**}$ weak* and $\limsup_d \|x + x^{**} - x_d\| \le \max(\|x\|, \|x^{**}\|)$.*

(3) *If A is a convex subset of X and $x^{**} \in X^{**}$ is in the weak*-closure of A then there is a sequence (x_n) in X so that $\lim_n x_n = x^{**}$ weak* and so that for every $x \in X$, $\limsup_n \|x + x^{**} - x_n\| \le \max(\|x\|, \|x^{**}\|)$.*

Proof. (1)⇒(3). Since X^* is separable this will follow from a diagonal argument if we establish that whenever $x \in X$, $\varepsilon > 0$, $m \in \mathbb{N}$ and $x_n \to x^{**}$ weak* then there exists $y \in \mathrm{co}\{x_m, x_{m+1}, \ldots\}$ such that $\|x + x^{**} - y\| < \max(\|x\|, \|x^{**}\|) + \varepsilon$. If this fails then by the Hahn–Banach theorem, there exists $x^{***} \in X^{***}$ with $\|x^{***}\| \le 1$ and $\Re x^{***}(x + x^{**} - x_n) \ge a + \varepsilon$, where $a = \max(\|x\|, \|x^{**}\|)$. Write $x^{***} = f + \phi$ where $f = \pi x^{***}$. Then

$$\Re\phi(x^{**}) + \Re f(x + x^{**} - x_n) \ge a + \varepsilon.$$

Letting $n \to \infty$ we have

$$\Re\phi(x^{**}) + \Re f(x) \geq a + \varepsilon .$$

Since $\|\phi\| + \|f\| = \|x^{***}\| \leq 1$ this leads to a contradiction.

(3)⇒(2). This is immediate.

(2)⇒(1). Suppose $f \in X^*$, $\phi \in X^\perp$. For $\varepsilon > 0$ we can find $x \in B_X$ so that $\Re f(x) > \|f\| - \varepsilon$ and $x^{**} \in X^{**}$ so that $\Re\phi(x^{**}) > \|\phi\| - \varepsilon$. Pick the net (x_d) in X so that $x_d \to x^{**}$ weak* and $\limsup \|x + x^{**} - x_d\| \leq 1$. Thus

$$\limsup_d \Re(\phi(x^{**}) + f(x^{**} - x_d + x)) \leq \|\phi + f\|$$

and thus

$$\|\phi\| + \|f\| - 2\varepsilon \leq \|\phi + f\| .$$

The conclusion follows immediately. ∎

PROPOSITION 4.4. *Let X be a separable M-ideal and suppose X has a boundedly complete Schauder decomposition $(E_n)_{n=1}^\infty$. Then all but finitely many of the E_n are reflexive.*

Proof. It suffices to show that we cannot have all E_n nonreflexive. Suppose every E_n is nonreflexive. Pick $e_n^{**} \in E_n^{\perp\perp}$ with $\|e_n^{**}\| = d(e_n^{**}, E_n) = 1$. This is possible since E_n is an M-ideal and hence proximinal. Now choose $e_{nk} \in E_n$ so that $\lim_k e_{nk} = e_n^{**}$ weak* and for every $x \in X$,

$$\limsup_k \|x + e_n^{**} - e_{nk}\| \leq \max(1, \|x\|) .$$

Suppose $\delta_n > 0$ for $n \geq 1$ and $\sum \delta_n < \infty$. We will pick a sequence (x_n) in E_n so that $\|x_n\| \geq 1/2$ but $\|s_n\| \leq C_n = \prod_{k=1}^n (1+\delta_k)$ where $s_n = \sum_{k=1}^n x_k$. Let $s_0 = 0$. Then for $n \geq 1$ if $\{x_j : j < n\}$ have been determined we note that

$$\limsup_k \|s_{n-1} + e_n^{**} - e_{nk}\| < C_n .$$

Hence there exists k so that

$$\|s_{n-1} + e_n^{**} - e_{nk}\| < C_n .$$

It follows that we can find a sequence (g_l) in E_n converging weak* to $e_n^{**} - e_{nk}$ so that $\|s_{n-1} + g_l\| < C_n$. Of course $\liminf_l \|g_l\| \geq d(e_n^{**}, E_n) = 1$. Thus we may pick x_n so that $x_n \in E_n$, $\|x_n\| \geq 1/2$ and $\|s_{n-1} + x_n\| < C_n$.

Now observe that $\sum_{k=1}^n x_k$ is bounded and not convergent, contradicting the fact that (E_n) is a boundedly complete decomposition. ∎

EXAMPLE (5). It is now possible to give an example of a space X which is a strict u-ideal but X has no equivalent norm in which it is an M-ideal. For we may take $X = \ell_2(c_0)$ which is a strict u-ideal (or a strict h-ideal in the complex case) by Example (1) above but cannot be renormed to be an M-ideal by Proposition 4.4.

EXAMPLE (6). We conclude this section by giving examples of strict u-ideals which fail important properties enjoyed by M-ideals.

First, we give an example of a u-ideal which is not proximinal. To do this equip c_0 with the norm $\|x\|_\infty + (\sum_{n=1}^\infty 2^{-n}|x_n|^2)^{1/2}$. This is a lattice norm so that c_0 is a u-ideal for this norm. However, if e is the constantly one sequence in ℓ_∞, then $d(e, c_0) = 1$ but the distance is not attained.

Second, consider the space $\mathbb{R} \oplus_1 c_0$. This is the space of real sequences $(x_n)_{n\geq 0}$ under the norm $|x_0| + \max_{n\geq 1} |x_n|$. Again we have a u-ideal norm. Here the functional $x \to x_0$ has a norm-preserving extension to ℓ_∞ given by $x \to x_0 + \lim_{\mathcal{U}} x_n$ where $\mathcal{U}$ is any nonprincipal ultrafilter on $\mathbb{N}$. Thus $\mathbb{R} \oplus_1 c_0$ is not Hahn–Banach smooth.

5. Strict u-ideals. In this section we consider strict u-ideals, i.e. Banach spaces X which are strict u-ideals in their biduals X^{**}. We remark that ℓ_1 is a u-ideal because it is in fact a u-summand in ℓ_1^{**}; it is therefore not a strict u-ideal (by applying Lemma 3.1). The following theorem extends this remark considerably.

THEOREM 5.1. *Suppose that X is a separable Banach space containing ℓ_1. Let P be a contractive projection on X^{***} with $\ker P = X^\perp$ and such that $V = P(X^{***})$ is norming. Then $\|I - P\| \geq 2$. In particular, X cannot be a strict u-ideal.*

Proof. Since V is norming the associated operator $T : X^{**} \to X^{**}$ is an isometry. If X contains a copy of ℓ_1, then by a theorem of Maurey [50] (see [37]), there exists $x^{**} \in X^{**}$ with $\|x^{**}\| = 1$ and such that $\|x^{**} + x\| = \|x^{**} - x\|$ for all $x \in X$. If $\|I - P\| = a$ then by Lemma 2.2 we can find a net (x_d) in X, converging weak* to Tx^{**}, with $\limsup \|Tx^{**} - x_d\| \leq a$. Since T is an isometry $\limsup \|x^{**} - x_d\| \leq a$ and thus $\limsup \|x^{**} + x_d\| = \limsup \|Tx^{**} + x_d\| \leq a$. However, $\limsup \|Tx^{**} + x_d\| \geq 2$. ∎

PROPOSITION 5.2. *Let X be either a separable Banach space or a Banach space containing no copy of ℓ_1.*

(1) *X is a strict u-ideal if and only if $\|I - 2\pi\| = 1$, i.e. if and only if $2 \in G(X)$.*

(2) *(If X is complex) X is a strict h-ideal in X^{**} if and only if $\|I - (1+\lambda)\pi\| = 1$ whenever $|\lambda| \leq 1$, i.e. if and only if $G(X) = \{1 + \lambda : |\lambda| \leq 1\}$.*

(3) *If X is a strict u-ideal (resp. strict h-ideal) then every subspace of a quotient space of X is also a strict u-ideal (resp. strict h-ideal).*

Proof. (1) and (2). Note first that if X is separable then Theorem 5.1 implies that X contains no copy of ℓ_1. These statements follow immediately if we show that V, the u-complement of $X^\perp$ in X^{***}, coincides with (the canonical image of) X^*. However, V is norming and it is shown in [26] (see [23] for the separable case) that under the hypothesis that X contains no copy of ℓ_1, X^* is contained in every norming subspace of X^{***}. Hence $X^* \subset V$ from which the result is immediate.

For (3) we need only appeal to Lemma 2.5. ∎

LEMMA 5.3. *Let X be a Banach space not containing ℓ_1. Then $\|I-2\pi\| \le \kappa_{\mathrm{u}}(X)$; further, if X is complex and $|\lambda-1|\le 1$ then $\|I-\lambda\pi\| \le \kappa_{\mathrm{h}}(X)$.*

Proof. By using condition (4) of Proposition 2.3 it is easy to see that it will suffice to prove this when X is separable. In that case suppose $x^{**} \in S_{X^{**}}$. Then, by [51], $x^{**} \in Ba(X)$ and so, for $\varepsilon > 0$, there is a series $\sum x_k = x^{**}$ weak* such that for all $-1 \le \theta_k \le 1$ and all n,

$$\Big\|\sum_{k=1}^{n}\theta_k x_k\Big\| \le \liminf_{m\to\infty}\Big\|\sum_{k=n+1}^{m} x_k - \sum_{k=1}^{n} x_k\Big\| \le \kappa_{\mathrm{u}}(X)+\varepsilon .$$

Hence if $s_n = \sum_{k=1}^{n} x_k$ then

$$\|x^{**}-2s_n\| \le \kappa_{\mathrm{u}}(x^{**})+\varepsilon .$$

By Proposition 2.3 again, $\|I-2\pi\| \le \kappa_{\mathrm{u}}(X)+\varepsilon$. The complex case is similar. ∎

Now we come to our main theorems characterizing spaces which are strict u-ideals (in their biduals). It was shown in [27] that if X is an M-ideal in X^{**} then X has property (u) with constant one, i.e. $\kappa_{\mathrm{u}}(X)=1$. The proof given there is quite different. (See [34] for the M-ideal analogue of Lemma 3.4.)

THEOREM 5.4. *Let X be a Banach space containing no copy of ℓ_1. Then X is a strict u-ideal if and only if $\kappa_{\mathrm{u}}(X)=1$ (i.e. X has property* (u) *with constant one). X is a strict h-ideal in X^{**} if and only if $\kappa_{\mathrm{h}}(X)=1$.*

Proof. Suppose X is a strict u-ideal (resp. strict h-ideal) in X^{**}. Then by Proposition 5.2, the projection $P=\pi$ and the associated operator $T : X^{**}\to X^{**}$ is the identity. Now if $x^{**}\in Ba(X)$ then we select a sequence (x_n) converging weak* to x^{**}. Now let A_n be the convex hull of $\{x_k : k\ge n\}$ and apply Lemma 3.4. If H_n is the weak*-closure of A_n then $\bigcap_n H_n = \{x^{**}\}$ so that we conclude that $\kappa_{\mathrm{u}}(x^{**}) = \|x^{**}\|$ (resp. $\kappa_{\mathrm{h}}(x^{**}) = \|x^{**}\|$).

For the other direction, Lemma 5.3 gives the implication. ∎

The next theorem gives an hereditary characterization of strict u-ideals.

THEOREM 5.5. *Let X be an arbitrary Banach space. Then the following conditions are equivalent*:

(1) *X does not contain a copy of ℓ_1 and X is a strict u-ideal.*

(2) *For every closed subspace Y of X and every $y^{**}\in Y^{**}$ with $\|y^{**}\|=1$,*

$$\inf_{y\in S_Y}\|y^{**}-2y\| = 1 .$$

Remark. If X is separable then it is not necessary to assume X contains no copy of ℓ_1 in (1).

Proof. The direction (1)⇒(2) is immediate from Proposition 2.3. We consider the converse; assume (2). First assume X is separable. We will show that X^* is separable. In fact, let N be a closed norming subspace of X^*. Then if $x^{**}\in N^{\perp}$ we have $\|x^{**}-x\|\ge\|x\|$ for all $x\in X$. In particular, $\inf_{x\in S_X}\|x^{**}-2x\|\ge 2$. We conclude that $N=X^*$ and the fact that X^* contains no proper norming subspaces implies that X^* is separable.

Thus X^* has the Radon–Nikodym Property and (see e.g. [6]), the norm on X^{**} is Fréchet-differentiable at each point of a norm-dense subset of $S_{X^{**}}$. Let us consider such a point x^{**}. Let (H_n) be an increasing sequence of finite-dimensional subspaces of X^* whose union is norm-dense in $\ker x^{**}$. For any n we can pick $x_n\in(H_n)_{\perp}\subset X$ so that $\|x_n\|=1$ and $\|x^{**}-2x_n\|\le 1+n^{-1}$.

Consider any weak*-cluster point χ of the sequence (x_n). Clearly $\chi\in(\ker x^{**})^{\perp}$ so that $\chi=\mu x^{**}$ for some μ and of course $|\mu|\le 1$. Pick $x_n^*\in S_{X^*}$ so that $x_n^*(x_n)=1$. Then

$$|\langle x_n^*, x^{**}-2x_n\rangle| \le 1+n^{-1}$$

and it follows that $\lim x^{**}(x_n^*)=1$. Since the norm is Fréchet-differentiable at x^{**} this implies by Shmulyan's lemma that $\lim\|x^*-x_n^*\|=0$ where $x^*\in S_{X^*}$ is the differential of the norm at this point. It follows that $\lim x^*(x_n)=1$. However, μ is a cluster point of the sequence $x^*(x_n)$ so $\mu=1$.

Our conclusion is that there is a sequence (x_n) converging weak* to x^{**} so that $\limsup\|x^{**}-2x_n\|=1$. By a density argument this holds for all $x^{**}\in S_{X^{**}}$ and an appeal to Proposition 2.3 shows that $\|I-2\pi\|=1$. Thus $2\in G(X)$ and by 5.2 we have (1).

In general if X is nonseparable, we observe that the above argument gives that every separable subspace Y satisfies $\kappa_{\mathrm{u}}(Y)=1$. This trivially implies $\kappa_{\mathrm{u}}(X)=1$ and so X is a strict u-ideal in X^{**} not containing ℓ_1. ∎

Remark. It follows from Proposition 2.7 and Lemma 2.5 that if X is a strict u-ideal in X^{**} (and contains no copy of ℓ_1 if X is nonseparable) then every subspace Y of a quotient of X has the property that every proper closed subspace of Y^* has characteristic at most 1/2. We do not know of any general converse to this result (see Section 10, Question 2).

Before our next result we prove two general results concerning isometries which use the ideas of [26] and will also be used in Section 6. We refer also to [24] and [29] for similar results.

LEMMA 5.6. *Let X be a Banach space and let $U : X^{**}\to X^{**}$ be an invertible isometry. Then*:

(1) *If X contains no copy of ℓ_1 then U is weak*-continuous.*

(2) *In general, if (x_n^{**}) is a weakly Cauchy sequence such that $\lim_{n\to\infty} x_n^{**} = x^{**}$ weak* then $\lim_{n\to\infty} Ux_n^{**} = Ux^{**}$ weak*.*

Proof. U is necessarily continuous for the ball topology $b_{X^{**}}$ on X^{**} (cf. [26]).

(1) By Corollary 5.5 of [26] if X contains no copy of ℓ_1 the ball topology on X^{**} coincides on $B_{X^{**}}$ with the weak*-topology and so U is weak*-continuous on $B_{X^{**}}$, which implies the result.

(2) In this case Ux_n^{**} is clearly weakly Cauchy and $\lim Ux_n^{**} = Ux^{**}$ for the ball topology $b_{X^{**}}$. Suppose $\lim Ux_n^{**} = \chi$ for the weak*-topology. Let A be the norm-closed absolutely convex hull of the set $\{Ux_n^{**} : n \in \mathbb{N}\} \cup \{Ux^{**}, \chi\}$. Then it is not difficult to show that A is weakly precompact, i.e. every sequence in A contains a weakly Cauchy subsequence. Thus A is a Rosenthal set (Definition 3.2 of [26]) and by Theorem 3.3 of [26], $b_{X^{**}}$ is Hausdorff when restricted to this set. Now since the sequence (Ux_n^{**}) converges to both Ux^{**} and to χ for the ball topology we have $\chi = Ux^{**}$ as required. ∎

In our next result (1) is proved for M-ideals in [35], by a similar method.

THEOREM 5.7. *Let X be a Banach space which contains no copy of ℓ_1, and is a strict u-ideal. Then:*

(1) *If $U : X^{**} \to X^{**}$ is an invertible isometry, there exists an isometry $U_0 : X \to X$ so that $U = U_0^{**}$.*

(2) *X is the unique isometric predual of X^* which is a strict u-ideal.*

Proof. (1) By Lemma 5.6, $U^*(X^*) = X^*$. Now $U^*(X^\perp)$ is a u-complement of X^* and so by Lemma 3.1, $U^*(X^\perp) = X^\perp$. This implies that $U(X) = X$ and that $U = U_0^{**}$ where U_0 is its restriction to X.

(2) By Lemma 3.1, $X^\perp$ is the unique u-complement of X^*. ∎

Remark. Thus ℓ_1, for example, has exactly one isometric predual X with $\kappa_u(X) = 1$, namely $X = c_0$. It is an open question whether an isomorphic predual X of ℓ_1 with $\kappa_u(X) < \infty$ is isomorphic to c_0 (see [21], [28], [54]). We will give some partial results in Section 7.

6. h-ideals. We now turn to the general theory of h-ideals, with no strictness assumptions. The theory of h-ideals is particularly simple because we are able to exploit the theory of hermitian operators on a general Banach space. We begin with a general result on such operators.

LEMMA 6.1. *Let X be an arbitrary complex Banach space. Let $H : X^{**} \to X^{**}$ be an hermitian operator such that $Hx = 0$ for $x \in X$.*

(1) *In general, $Hx^{**} = 0$ for $x^{**} \in Ba(X)$.*

(2) *If X contains no copy of ℓ_1 then $H = 0$.*

Proof. For all real t, the operator $\exp(itH)$ is an invertible isometry on X^{**} which satisfies $\exp(itH)x = x$ for $x \in X$. In case (1), Lemma 5.6(2) gives that $\exp(itH)x^{**} = x^{**}$ for all $x^{**} \in Ba(X)$. It follows that $Hx^{**} = 0$. In case (2), $\exp(itH)$ is weak*-continuous and hence coincides with the identity for all t, whence $H = 0$. ∎

Let us now suppose that X is an h-ideal and that P is the associated hermitian projection on X^{***} with $\ker P = X^\perp$. Let $T : X^{**} \to X^{**}$ be defined as usual by the formula $\langle x^*, Tx^{**}\rangle = \langle x^{**}, Px^*\rangle$, for $x^* \in X^*$ and $x^{**} \in X^{**}$.

LEMMA 6.2. *T is hermitian on X^{**} and $Tx = x$ for $x \in X$.*

Proof. By Theorem 2, p. 11 of [5] it suffices to show that if $x^* \in S_{X^*}$, $x^{**} \in S_{X^{**}}$ and $x^{**}(x^*) = 1$ then $\langle x^*, Tx^{**}\rangle$ is real. But this is immediate since P is hermitian. The last statement has already been noted. ∎

LEMMA 6.3. *Let X be an arbitrary real or complex Banach space and suppose $x^{**} \in S_{X^{**}}$ satisfies $\kappa_u(x^{**}) < 2$. Then $\ker x^{**}$ cannot be a norming subspace of X^*.*

Proof. Let $x^{**} = \sum x_n$ weak* where if $\theta_k = \pm 1$ then $\|\sum_{k=1}^n \theta_k x_k\| \le 2 - \delta$ for some $\delta > 0$. If $s_n = \sum_{k=1}^n x_k$ then $\|x^{**} - 2s_n\| \le 2 - \delta$. Then there is a sequence of convex combinations t_n of s_n so that t_n converges weak* to x^{**} and $\lim \|t_n\| = 1$. Thus $\|x^{**} - 2t_n\| \le 2 - \delta$, which leads to the fact that if $x^* \in \ker x^{**}$ and $\|x^*\| = 1$ then $|x^*(t_n)| \le 1 - \delta/2$. Thus $\ker x^{**}$ cannot be norming.

Remark. If X is a separable Banach space containing no copy of ℓ_1 and such that $\kappa_u(X) < 2$ then this lemma yields that X^* is also separable since it has no proper norming subspace. We note that the James Hagler space JH is a space with property (u) [46] but JH^* is nonseparable; thus $\kappa_u(JH) \ge 2$ for any renorming.

THEOREM 6.4. *Let X be an h-ideal. Then $\kappa_h(X) = 1$.*

Proof. By Lemmas 6.1 and 6.2 we have $Tx^{**} = x^{**}$ for $x^{**} \in Ba(X)$. Hence by Lemma 3.4, $\kappa_h(x^{**}) \le \|x^{**}\|$ and so $\kappa_h(X) = 1$. ∎

Of course the condition $\kappa_h(X) = 1$ is not sufficient for X to be an h-ideal. In the case when X is separable, we can, however, give a fairly complete description of h-ideals.

THEOREM 6.5. *Let X be a separable complex Banach space. Then in order that X be an h-ideal it is necessary and sufficient that there exists an hermitian projection T of X^{**} onto $Ba(X)$ such that for every $x^{**} \in X^{**}$*

with $\|x^{**}\| = 1$ *there is a net (sequence)* (x_d) *in* X *so that* $\lim_d x_d = Tx^{**}$ *weak* and for every* $|\lambda| = 1$, $\limsup_d \|x^{**} - (1+\lambda)x_d\| \le 1$.

Proof. First assume that X is an h-ideal. In this case the conclusion will be immediate from Lemmas 2.2 and 6.2 once we show that T is a projection onto $Ba(X)$. As observed in Theorem 6.4, $Tx^{**} = x^{**}$ for $x^{**} \in Ba(X)$.

We first claim that if $x^{**} \in Ba(X)$ is such that ker x^{**} is norming then $x^{**} = 0$. This follows immediately from the fact that $\kappa_h(x^{**}) = \|x^{**}\|$, and Lemma 6.3.

Now if X is separable there is a separable norming subspace M in X^*. By Lemma 3.4 if $x^{**} \in S_{X^{**}}$ and $\varepsilon > 0$ we can find $\chi \in Ba(X)$ with $\kappa_h(\chi) < 1 + \varepsilon$ and such that $\chi(f) = Tx^{**}(f)$ for $f \in M$. Now suppose x^* is an arbitrary element of X^*. Then there exists $\chi_1 \in Ba(X)$ so that $\chi_1(f) = Tx^{**}(f)$ for $f \in M$ and $\chi_1(x^*) = Tx^{**}(x^*)$; this follows by applying the same argument to the span of M and x^*. Thus $\chi_1 - \chi \in Ba(X)$ and $M \subset \ker(\chi_1 - \chi)$. It follows that $\chi_1 = \chi$ and hence that $Tx^{**}(x^*) = \chi(x^*)$ for all $x^* \in X^*$. Thus $Tx^{**} = \chi \in Ba(X)$. Hence T is an hermitian projection onto $Ba(X)$.

Now consider the converse direction. In this case we can define $P : X^{***} \to X^{***}$ by $P = T^*\pi$. Clearly πT^* is a projection of X^{***} onto X^* and so P is a projection whose kernel is $X^\perp$. It is now possible to use Lemma 2.2 to show P is indeed hermitian, since T coincides with the operator T induced from P as in Section 2. By hypothesis for each $x^{**} \in S_{X^{**}}$ there is a net (x_d) with $\lim x_d = Tx^{**}$ weak* and so that if $|\lambda| = 1$ then $\limsup \|x^{**} - (1+\lambda)x_d\| \le 1$. By Lemma 2.2 it follows that $\|I - (1+\lambda)P\| = 1$ if $|\lambda| = 1$ and this shows that $\|\exp(itP)\| = 1$ for all real t, i.e. P is hermitian. ∎

Remark. Thus if X is a separable h-ideal then we can conclude both that there is an hermitian projection of X^{**} onto $Ba(X)$ and that $\kappa_h(X) = 1$, i.e. X has complex property (u) with constant one. The converse is false but we postpone giving a counterexample until Section 8. Theorem 6.5 instead requires some interaction between property (u) and the projection T.

Next we characterize strict h-ideals.

THEOREM 6.6. *Let X be an h-ideal. Then the following are equivalent:*

(1) *X is a strict h-ideal.*
(2) *X^* is an h-ideal.*
(3) $\|I - \lambda\pi\| \le 1$ *if* $|\lambda - 1| \le 1$.
(4) *Every separable subspace of X has separable dual.*
(5) *X contains no copy of ℓ_1.*

Proof. (1)⇒(2). (Note this is immediate from Proposition 5.2 if X is separable.) In this case the operator $T : X^{**} \to X^{**}$ is an isometry. Since T is hermitian it follows that ([4]) $T(X) = T^2(X)$ and so T is invertible on $T(X)$. This implies that T is surjective and so its spectrum is contained in the unit circle. Since it is hermitian $\sigma(T) \subset \{\pm 1\}$. However, $\|I - (1+\lambda)T\| = 1$ for $|\lambda| = 1$ and so it follows that the spectrum of T is reduced to $\{1\}$. Thus the spectrum of $T - I$ reduces to $\{0\}$ and since $T - I$ is hermitian it follows from Sinclair's theorem ([4, p. 73], [59]) that $T - I = 0$. Hence $T = I$ and $P = \pi$ so that X^* is an h-summand in X^{***}.

(2)⇒(3). We argue first that X^* can contain no copy of c_0. Indeed, if so, then since it is a dual space ℓ_∞ embeds into X^* and so ℓ_∞ has property (u); this is clearly false. We conclude that X^* is an h-summand in X^{***} (Theorem 3.5). Let $Q : X^{***} \to X^*$ be an hermitian projection. Let $P : X^{***} \to V$ be the hermitian projection associated with the fact that X is an h-ideal. Then $i(PQ - QP)$ is hermitian ([4, p. 47]).

We now use the fact (cf. [55, Corollary to Theorem 2]) that if T is an hermitian operator on a space Y and if R is a norm one projection, then RT is hermitian on the space $R(Y)$. It follows that $i\pi(PQ - QP)$ is hermitian on X^*. However, $\pi P = \pi$ and $\pi Q = Q$. Hence $i(I_{X^*} - QP)$ is hermitian on X^*. Note that Q is also a norm-one projection onto X^* and so QP is hermitian on X^*. Hence $I_{X^*} - QP$ is hermitian. This implies $I_{X^*} - QP = 0$ on X^* and thus QP is another contractive projection onto X^*. Hence PQ is a contractive projection.

Now $PQ(PQ - QP)QP = PQ(PQ - QP)PQ = 0$ etc. so that $(PQ - QP)^3 = 0$. Since $i(PQ - QP)$ is hermitian this implies (again by Sinclair's theorem, or other more elementary arguments) that $PQ = QP$. Hence $X^* = V$ and $P = \pi$.

(3)⇒(4). This follows by Lemma 2.5 and Proposition 2.8.

(4)⇒(5). Immediate.

(5)⇒(1). Here Lemmas 6.1 and 6.2 imply that T is the identity on X^{**} and so $P = \pi$ and X is a strict h-ideal. ∎

EXAMPLE. Suppose X is an h-ideal with the property that X^{**} is isometric to a von Neumann algebra. Then X^* is an L-summand (see [61]). Thus the above theorem yields that X is a strict h-ideal and moreover X^* has the Radon–Nikodym Property (since every separable subspace of X has separable dual). Thus ([20, Theorem VII.8), X^* is isometric to an ℓ_1-sum of projective tensor products of Hilbert spaces. It follows from Theorem 5.7 that X must be isometric to a space $(\sum_{i\in I} \oplus \mathcal{K}(H_i))_{c_0}$.

We now consider the problem of identifying h-ideals, by first considering subspaces.

THEOREM 6.7. *Let X be a separable h-ideal and T be the induced hermitian projection of X^{**} onto $Ba(X)$. If Z is a subspace of X such that $Z^{\perp\perp}$ is T-invariant then Z is an h-ideal.*

Proof. Of course $Z^{\perp\perp}$ can be identified with Z^{**} and T restricts to an hermitian projection on Z^{**} whose range includes Z. If $z^{**} \in Z^{**}$ with $\|z^{**}\| = 1$ then Tz^{**} is in the weak*-closure of B_Z and so by use of Lemma 2.2 there is a net (z_d) in Z with $\lim z_d = Tz^{**}$ weak* and $\limsup \|z^{**} - (1+\lambda)z_d\| \le 1$ whenever $|\lambda| = 1$. The result follows by Theorem 6.5. ∎

We will prove a converse to this result. First we introduce a definition. We will say that a separable h-ideal is *nondegenerate* if whenever $\chi \in \ker T$ and $x^{**} \in Ba(X)$ then $\|\chi + x^{**}\| = \|\chi\|$ implies that $x^{**} = 0$ (i.e. $Ba(X)$ is a Chebyshev subspace of X^{**}). The simplest example of a degenerate h-ideal is $\mathbb{C} \oplus_\infty \ell_1$; note that in this case $X = Ba(X)$.

PROPOSITION 6.8. *Let X be a separable h-ideal and suppose $\chi \in S_{X^{**}}$ satisfies $T\chi = 0$. Suppose E is a finite-dimensional subspace of $Ba(X)$. Then, whenever (A_n) is a sequence of convex subsets of X so that χ is in the weak*-closure of each A_n and $\varepsilon > 0$ then there is a sequence (x_n) in A_n such that for any $x^{**} \in E$, and any complex $(\alpha_k)_{k=1}^n$ and β,*

$$\Big\|x^{**} + \sum_{k=1}^{n} \alpha_k x_k + \beta\chi\Big\| \ge (1-\varepsilon)\Big\|x^{**} + \Big(\sum_{k=1}^{n} |\alpha_k| + |\beta|\Big)\chi\Big\|$$

and, in particular,

$$\Big\|\sum_{k=1}^{n} \alpha_k x_k + \beta\chi\Big\| \ge (1-\varepsilon)\Big(\sum_{k=1}^{n} |\alpha_k| + |\beta|\Big).$$

Proof. This is essentially a rewording of an argument of Maurey [50]. We suppose $\delta_n > 0$ are chosen so that $\prod(1-\delta_n) = 1-\varepsilon$.

We select $x_n \in A_n$ by induction. To start the induction let $F_0 = E$ and then set $F_n = [E, x_1, \ldots, x_n]$. We choose $x_n \in A_n$ so that

$$\|f + \alpha x_n + \beta\chi\| \ge (1-\delta_n)(\|f + (|\alpha| + |\beta|)\chi\|)$$

whenever $f \in F_{n-1}$ and $\alpha, \beta \in \mathbb{C}$.

We now describe the selection of x_n. Let $G_n = [\chi, F_{n-1}]$, and then choose a finite set $\{x_1^*, \ldots, x_N^*\}$ in B_{X^*} so that if $g \in G_n$ then

$$\max_{1\le k\le N} |x_k^*(g)| \ge (1 - \tfrac{1}{2}\delta_n)\|g\|.$$

Pick $x_n \in A_n$ so that for $1 \le k \le N$ we have $|x_k^*(x_n) - \chi(x_k^*)| \le \frac{1}{2}\delta_n$. Now for $f \in F_{n-1}$, $\alpha \ge 0$, $\beta \in \mathbb{C}$ we have

$$\begin{aligned}\|f + \alpha x_n + \beta\chi\| &= \|f + \alpha x_n + |\beta|\chi\|\\ &\ge \max_{1\le k\le N} |x_k^*(f) + \alpha x_k^*(x_n) + |\beta|\chi(x_k^*)|\\ &\ge \max_{1\le k\le N} |x_k^*(f) + (|\alpha| + |\beta|)\chi(x_k^*)| - \tfrac{1}{2}|\alpha|\delta_n\end{aligned}$$

$$\begin{aligned}&\ge (1 - \tfrac{1}{2}\delta_n)\|f + (|\alpha| + |\beta|)\chi\| - \tfrac{1}{2}\delta_n|\alpha|\\ &\ge (1-\delta_n)\|f + (|\alpha| + |\beta|)\chi\|\end{aligned}$$

and this inequality must hold for all complex α by homogeneity.

Now it follows quickly by induction that

$$\Big\|x^{**} + \sum_{k=1}^{n} \alpha_k x_k + \beta\chi\Big\| \ge \prod_{k=1}^{n}(1-\delta_k)\Big\|x^{**} + \Big(\sum_{k=1}^{n} |\alpha_k| + |\beta|\Big)\chi\Big\|$$

for all $x^{**} \in E$, $\alpha_1, \ldots, \alpha_n, \beta \in \mathbb{C}$ and the result follows. ∎

THEOREM 6.9. *Let X be a separable nondegenerate h-ideal. Then a closed subspace Z of X is an h-ideal if and only if $Z^{\perp\perp}$ is T-invariant.*

Proof. We need only consider the case when Z is an h-ideal. Let $T_Z : Z^{**} \to Ba(Z)$ be the associated hermitian projection. In this case the ℓ_1-sum $X \oplus_1 Z$ is also an h-ideal, and the associated projection of $X^{**} \oplus_1 Z^{**}$ onto $Ba(X) \oplus_1 Ba(Z)$ is given by $T \oplus T_Z$.

Suppose $\chi \in Z^{**}$ satisfies $T_Z\chi = 0$ and $\|\chi\| = 1$. Identifying Z^{**} with $Z^{\perp\perp} \subset X^{**}$ in the natural way we may consider χ in X^{**}. Then $T\chi \in Ba(X)$ and so there is a sequence (u_n) in X converging weak* to $T\chi$. Let $\xi = \chi - T\chi$ (note here that $\chi = T\chi$ is impossible since $T\chi \in Ba(X)$ but $\chi \notin Ba(Z)$). Let $C_n \subset X \oplus_1 Z$ be the set of all (x, z) such that $z - x \in \operatorname{co}\{u_k : k \ge n\}$. Then (ξ, χ) is in the weak*-closure of each C_n.

If $\delta > 0$ then (ξ, χ) is also in the weak*-closure of $A_n = \{(x, z) \in C_n : \|x\| \le (1+\delta)\|\xi\|,\ \|z\| \le 1+\delta\}$. In fact, if $B = \{(x, z) : \|x\| \le \|\xi\|,\ \|z\| \le 1\}$ then for any weak*-neighborhood W of (ξ, χ), 0 is in the weak-closure of $(W \cap C_n) - B$ and hence also in the norm-closure. Hence $0 \in (W \cap C_n) - (1+\delta)B$, whence $W \cap C_n \cap (1+\delta)B$ is nonempty.

It follows that we can pick $(x_n, z_n) \in A_n$ so that for all scalars $\alpha_1, \ldots, \alpha_n$ and all $n \in \mathbb{N}$,

$$\Big\|\sum_{k=1}^{n} \alpha_k x_k\Big\| + \Big\|\sum_{k=1}^{n} \alpha_k z_k\Big\| \ge (1-\delta)(1 + \|\xi\|)\sum_{k=1}^{n} |\alpha_k|.$$

By construction $\lim(z_n - x_n) = T\chi$ weak* and so $\lim(z_{2n} - z_{2n+1} - x_{2n} + x_{2n+1}) = 0$ weakly. Thus there exist $n \in \mathbb{N}$, $\beta_k \ge 0$ such that $\sum \beta_k = 1$ and

$$\Big\|\sum_{k=1}^{n} \beta_k(z_{2k} - z_{2k+1}) - \sum_{k=1}^{n} \beta_k(x_{2k} - x_{2k+1})\Big\| \le \delta.$$

It follows that

$$\Big\|\sum_{k=1}^{n} \beta_k(z_{2k} - z_{2k+1})\Big\| \le 2(1+\delta)\|\xi\| + \delta,$$

and hence that

$$\Big\|\sum_{k=1}^{n}\beta_k(z_{2k}-z_{2k+1})\Big\|+\Big\|\sum_{k=1}^{n}\beta_k(x_{2k}-x_{2k+1})\Big\|\le 4(1+\delta)\|\xi\|+\delta.$$

Hence

$$2(1-\delta)(1+\|\xi\|)\le 4(1+\delta)\|\xi\|+\delta$$

and as $\delta>0$ is arbitrary this implies that $\|\xi\|\ge 1$. Hence $\|\xi\|=1$ and as $\xi+T\chi=\chi$ and X is a nondegenerate h-ideal this in turn implies that $T\chi=0$.

It now immediately follows that $T_Z\chi=T\chi$ for any $\chi\in Z^{\perp\perp}$ and so $Z^{\perp\perp}$ is T-invariant. ∎

COROLLARY 6.10. *Let X be a separable complex M-ideal. Then a closed subspace Z of X^* is an h-ideal if and only if it is weak*-closed.*

Proof. In this case X^* is also separable and is an L-summand. Thus it is, in particular, a nondegenerate h-ideal and the operator T coincides with the canonical projection π of X^{***} on X^*. A subspace Z is π-invariant if and only if it is weak*-closed. ∎

The above corollary is false if we assume X is merely a strict h-ideal; similarly the preceding theorem fails for degenerate h-ideals. In fact, let $X=\mathbb{C}\oplus_1 c_0$ so that $X^*=\mathbb{C}\oplus_\infty \ell_1$. Let $\chi\in X^{**}=\mathbb{C}\oplus_1\ell_\infty$ be the element $(1,e)$ where e is the constantly one sequence. Then $\ker\chi$ is isometric to ℓ_1 and hence is an h-ideal. In this case X^* is a degenerate h-ideal (actually h-summand). Notice also that $\ker\chi$ is a summand (contractively complemented) subspace of X^*; in fact, if Z is an h-ideal and Z^* is a nondegenerate h-summand then any summand of Z^* is weak*-closed.

COROLLARY 6.11. *Let X be a separable complex Banach space such that X^* is an L-summand. Let Y be a quotient of X. Then Y is an M-ideal if and only if Y is a strict h-ideal.*

Proof. Since X^* is an L-summand, Y^* is also an L-summand (Theorems 6.6 and 6.9). But if Y is also a strict h-ideal this means that $\pi_Y: Y^{***}\to Y^*$ is an L-projection by Lemma 3.1. ∎

Remark. This corollary applies in particular when X is a C^*-algebra.

We now turn to some examples. The simplest and most natural example of an h-ideal is a (complex) order-continuous Banach lattice X. In this case $X^\perp$ is a band in X^{***}. If we assume X is separable then $Ba(X)$ coincides with the band generated by X in X^{**} and T is the natural band projection (see [56]).

Now suppose (Ω,Σ,μ) is a σ-finite measure space and that X is a separable order-continuous Köthe function space on (Ω,μ) (cf. [49]). By the above remarks X is an h-ideal.

PROPOSITION 6.12. *Let X be a separable order-continuous Köthe function space on (Ω,μ). Let Z be a subspace of X. Consider the properties:*

(1) If (f_n) is a sequence in B_Z which converges μ-a.e. then there is a weakly Cauchy sequence (g_n) in B_Z such that $\lim(f_n-g_n)=0$ μ-a.e.

(2) Z is an h-ideal.

Then (1)⇒(2) and, if X is a nondegenerate h-ideal, (2)⇒(1).

Proof. We first make some observations about X. We can identify X^* with a Köthe function space on (Ω,μ) and further $Ba(X)$ can be identified with the Köthe function space $X_{\max}$ consisting of all measurable f so that

$$\|f\|_{X_{\max}}=\sup\{\|g\|_X: g\in X,\ |g|\le|f|\}<\infty.$$

We may select a weak order unit u for X—then $u>0$ a.e. Similarly, there exists $v\in X^*$ so that $v>0$ a.e. The F-norm

$$|\!|\!|f|\!|\!|=\int v\min(|f|,u)\,d\mu$$

defines the topology of convergence in measure.

We now claim that if $x^{**}\in X^{**}$ and $\varepsilon>0$ then there is a convex subset $A_\varepsilon(x^{**})$ of the set $N_\varepsilon(Tx^{**})=\{f\in X: |\!|\!|f-Tx^{**}|\!|\!|\le\varepsilon\}$ such that x^{**} is in the weak*-closure of $A_\varepsilon(x^{**})$. This is clear if $x^{**}\in Ba(X)=X_{\max}$, since there is a sequence (f_n) in X so that f_n converges both weak* and almost everywhere to x^{**}; thus we can take $A_\varepsilon(x^{**})$ to be one of the sets $\mathrm{co}\{f_k: k\ge n\}$. We next suppose $x^{**}\ge 0$ and $Tx^{**}=0$; then x^{**} is in the weak*-closure of the set $A_W=\{f\in X: f\ge 0, f\in W\}$ for any weak*-neighborhood W of x^{**}. We claim that for suitable W we have $A_W\subset N_\varepsilon(0)$. Indeed, if not there is a net (f_d) converging weak* to x^{**} with $f_d\ge 0$ but $|\!|\!|f_d|\!|\!|>\varepsilon$. But then $\min(f_d,u)$ is contained in a weakly compact set and any limit point g satisfies $0\le g\le x^{**}$; as $g\in X$ this implies $g=0$ by the definition of the band projection, and hence $\int v\min(f_d,u)\,d\mu$ converges to zero, contrary to assumption. Now any general x^{**} can be written as $Tx^{**}+\chi_1-\chi_2$ where $T\chi_1=T\chi_2=0$ and $\chi_1,\chi_2\ge 0$; it is then clear that by adding sets we obtain the general claim.

Finally, our theorem is proved if we can show that $Z^{\perp\perp}$ is T-invariant if and only condition (1) holds. First suppose (1) holds. Then suppose $z^{**}\in Z^{\perp\perp}$; let $\|z^{**}\|=1$. Then, for $\varepsilon>0$, z^{**} is in the weak*-closure of $A_\varepsilon(Tz^{**})$ and by a Hahn–Banach argument there exist $z\in B_Z$ and $f\in A_\varepsilon(Tz^{**})$ with $\|z-f\|<\varepsilon$. Thus there is a sequence (f_n) in B_Z converging in measure to Tz^{**}. By passing to a subsequence we may assume f_n converges a.e. and thus by (1) there is a similar sequence (g_n) which is weakly Cauchy. Thus $Tz^{**} \in Z^{\perp\perp}$.

Conversely, assume $Z^{\perp\perp}$ is T-invariant, and assume $f_n \in B_Z$ converges a.e. to $\chi \in Ba(X)$. We may assume that $\|\chi - f_n\| \le 2^{-n}$. Let $A_n = \mathrm{co}\{f_k : k > n\}$; then for $f \in A_n$ we have $\|f - \chi\| \le 2^{-n}$. Now let z^{**} be any point in the weak*-closure of every A_n. Then $Tz^{**} \in Z^{\perp\perp}$. However, for $\varepsilon > 0$ and $n \in \mathbb{N}$, we can find, again by a Hahn–Banach argument, $z \in A_\varepsilon(Tz^{**})$ and $f \in A_n$ with $\|z - f\| < \varepsilon$. Thus $Tz^{**} = \chi$ and so $\chi \in Ba(X) \cap Z^{\perp\perp}$; it now follows that there exists a sequence (g_n) in B_Z converging a.e. and weak* to χ. ∎

Remark. If, in the above theorem, X is weakly sequentially complete then (1) is equivalent to the statement that B_Z is closed for the topology of convergence in measure.

This has immediate connections with the notion of a nicely placed subspace of L_1 (cf. [25]).

COROLLARY 6.13. *A closed subspace of $L_1[0,1]$ is an h-ideal if and only if it is nicely placed.*

We remark here that the real version of Corollary 6.13 is also true and can be proved by direct arguments; a closed subspace of $L_1[0,1]$ is a u-ideal if and only if it is nicely placed.

We also remark without proof that in the nonseparable case X is an h-ideal so that X^* is a nondegenerate h-summand then X is weakly compactly generated. This follows by arguments similar to those in [15].

7. u-ideals. In the case of u-ideals (and, in particular, for real Banach spaces) the analogous results are not quite so clear. We first consider the case when X contains no subspace isomorphic to ℓ_1. In this case one expects that a u-ideal is strict, but we only obtain a much weaker conclusion, unless we impose additional hypotheses.

PROPOSITION 7.1. *Let X be a Banach space containing no copy of ℓ_1 which is a u-ideal. Then V is weak*-dense in X^{***}.*

Proof. This amounts to proving that T is one-one. Suppose then that $x^{**} \in X^{**} \setminus \{0\}$ and $Tx^{**} = 0$. Notice that since $\|I - 2P\| \le 1$ it follows that $\|I - 2T\| \le 1$. Hence if $x \in X$ we have $\|x^{**} - x\| \le \|x^{**} + x\|$. This is impossible by Maurey's theorem [50] (see also [37]). ∎

PROPOSITION 7.2. *Let X be a Banach space containing no copy of ℓ_1. Suppose P is a projection on X^{***} such that $\ker P = X^\perp$ and $\|P\| = 1$. Let $V = P(X^{***})$. Then $V \cap X^*$ is norming for X.*

Proof. We consider again the associated map T. It is clear from the definition that if $x^* \in X^*$ and $T^*x^* = x^*$ then $x^* \in V$. Now for each $\chi \in X^{**}$ we consider the set $E_\chi = \{x^* \in X^* : T\chi(x^*) = \chi(x^*)\}$.

Now χ is of the first Baire class on (B_{X^*}, w^*) ([36], [51]) and therefore the set $\mathcal{C}_*(\chi)$ of points of continuity is a dense G_δ-set. Assume $x^* \in \mathcal{C}_*(\chi)$. Let $v = Px^*$. Then there is a net (x_d^*) in B_{X^*} converging in the weak*-topology of X^{***} (i.e. $\sigma(X^{***}, X^{**})$) to v. However, $v - x^* \in X^\perp$ so that x_d^* converges for $\sigma(X^*, X)$ to x^*. Thus

$$v(\chi) = \lim_d \chi(x_d^*) = \chi(x^*)$$

so that $Px^*(\chi) = \chi(x^*)$. Now $T\chi(x^*) = Px^*(\chi)$ so we conclude that $x^* \in E_\chi$. Hence E_χ is norming and further by [26] (or [23]) this implies that $H = \bigcap_{\chi \in X^{**}} E_\chi$ is also norming. However, $H \subset V \cap X^*$. ∎

PROPOSITION 7.3. *Under the hypotheses of Proposition 7.2, if moreover V r-norms X^{**} and $\|I - \lambda P\| = 1$ where $r\lambda > 1$ then $V = X^*$.*

Proof. Suppose $\|x^{**}\| = 1$ and that $x^{**} \in (V \cap X^*)^\perp$. Then it is easy to see that $Tx^{**} \in (V \cap X^*)^\perp$. Thus $\ker Tx^{**}$ norms X and so for $x \in X$ we have $\|Tx^{**} + x\| \ge \|x\|$. Now by Lemma 2.2 there is a net (x_d) converging weak* to Tx^{**}, with $\limsup \|Tx^{**} - \lambda x_d\| \le 1$. Hence $\limsup \|x_d\| \le \lambda^{-1}$ and so $\|Tx^{**}\| \le \lambda^{-1}$. However, $\|Tx^{**}\| \ge \sup_{v \in B_V} |v(x^{**})| \ge r$. This contradiction shows that $V \cap X^* = X^*$ and the result follows. ∎

THEOREM 7.4. *Let X be a Banach space containing no copy of ℓ_1. Suppose X is a u-ideal. Then the following conditions are equivalent:*

(1) *X^* contains no proper norming subspace.*
(2) *V r-norms X^{**} for some $r > 1/2$.*
(3) $\|I - 2\pi\| < 2$.
(4) $\kappa_{\mathrm{u}}(X) < 2$.
(5) *X is a strict u-ideal.*

Proof. (1)⇒(5). $V = X^*$ by Proposition 7.2. The converse is immediate (see the remark after Theorem 5.5).

(2)⇒(5). We need only apply Proposition 7.3. The converse is obvious.

(3)⇒(1) by Proposition 2.7 and (4)⇒(3) by Lemma 5.3. (5)⇒(4) is Theorem 5.4. ∎

We now consider the general case of separable u-ideals and prove a result analogous to but much weaker than Theorem 6.5.

THEOREM 7.5. *Let X be a separable u-ideal such that $\kappa_{\mathrm{u}}(X) < 2$. Then $\kappa_{\mathrm{u}}(X) = 1$ and $Ba(X)$ is a u-summand in X^{**}.*

Proof. First notice that by Lemma 6.3, if $x^{**} \in Ba(X)$ is such that $\ker x^{**}$ is norming then $x^{**} = 0$.

Suppose $x^{**} \in X^{**}$. Let M be a separable norming subspace of X^*. Suppose $\varepsilon > 0$. Then by Lemma 3.4 there exists $\chi \in X^{**}$ with $\kappa_u(\chi) \le \|x^{**}\| + \varepsilon$ so that $\chi(f) = Tx^{**}(f)$ for all $f \in M$. Arguing exactly as in Theorem 6.5 this leads to the conclusion that $\chi = Tx^{**}$. Thus T maps X^{**} into $Ba(X)$.

Next if $x^{**} \in Ba(X)$, then the argument of Proposition 7.2 shows that the set of $x^* \in B_{X^*}$ such that $x^{**}(x^*) = Px^*(x^{**})$ contains a weak*-dense G_δ-subset. Hence $x^{**} - Tx^{**}$ vanishes on a norming subspace of X^* and is in $Ba(X)$. Hence $Tx^{**} = x^{**}$. Thus T is a projection of X^{**} onto $Ba(X)$ and of course $\|I - 2T\| = 1$. Further it follows that if $x^{**} \in Ba(X)$ then $\kappa_u(x^{**}) = \|x^{**}\|$. ■

We conclude this section by considering a special result for M-ideals. We first need an adaptation of Theorem 5 of [2] (see also [13, Theorem 2.6] and [12]).

PROPOSITION 7.6. *Let Z be a Banach space and let J be an M-ideal in E. Let Y be a separable Banach space with (MAP). Then for any bounded operator $S : Y \to E/J$ there exists an operator $\widetilde{S} : Y \to E$ so that if $q : E \to E/J$ is the quotient map then $q\widetilde{S} = S$ and $\|\widetilde{S}\| = \|S\|$.*

THEOREM 7.7. *Let X be a separable M-ideal (in X^{**}) with (MAP). Let Y be a Banach space such that Y^* is isomorphic to X^*. Suppose that $dk - 1 < 2d^{-1}$ where $d = d(X^*, Y^*)$ and $k = \kappa_u(Y)$. Then X is isomorphic to Y. In particular, X is isomorphic to Y if either*

(1) *X^* is isometric to Y^* and $\kappa_u(Y) < 3$, or*
(2) *$d(X^*, Y^*) < 2$ and $\kappa_u(Y) = 1$ (i.e. Y is a strict u-ideal).*

Proof. We will suppose that $\phi : X^* \to Y^*$ is an isomorphism which satisfies $a^{-1}\|x^*\| \le \|\phi x^*\| \le \|x^*\|$ for $x^* \in X^*$, where $0 < a < \infty$. We will also suppose that the canonical projection $\pi_Y : Y^{***} \to Y^*$ satisfies $\|I - 2\pi_Y\| = b$. Let $q : X^{**} \to X^{**}/X$ be the quotient map. Let S be the restriction of ϕ^* to Y. Thus $S : Y \to X^{**}$ satisfies $a^{-1}\|y\| \le \|Sy\| \le \|y\|$.

Let us suppose $\|qS\| < a^{-1}$. We note first that by [31], X^* has (AP) and so Y has (MAP). Thus by Proposition 7.6 there is an operator $S_0 : Y \to X^{**}$ such that $\|S_0\| < a^{-1}$ and $qS_0 = qS$. Now $S - S_0$ maps Y into X and is an isomorphism onto its range. We claim $S - S_0$ is onto X. Indeed, if $x^* \in X^*$ with $\|x^*\| = 1$ and $x^*(S - S_0)(y) = 0$ for all $y \in Y$ then $|Sy(x^*)| \le \|S_0\|\|y\|$ so that $|\phi(x^*)(y)| \le \|S_0\|\|y\|$ for all $y \in Y$. This implies $\|\phi x^*\| < a^{-1}$, which is a contradiction. Thus the theorem will be established if we show that $\|qS\| < a^{-1}$.

Suppose $y \in S_Y$. Select $\xi \in X^{\perp\perp}$ with $\|\xi\| = 1$ and $\langle Sy, \xi\rangle = \|qSy\|$.

Then
$$\|qSy\| = \langle \phi^* y, \xi\rangle = \langle y, \phi^{**}\xi\rangle = \langle y, \pi_Y\phi^{**}\xi\rangle \le \|\pi_Y\phi^{**}\xi\| \le \|Q\xi\|,$$
where $Q = \phi^{-1}\pi_Y\phi^{**}$ is a projection of X^{***} onto X^*. Thus since X is an M-ideal we have
$$\|qSy\| \le a(\|(I + Q - \pi_X)\xi\| - 1) \le a^{-1}(\|I + Q - \pi_X\| - 1).$$
Now
$$\|I + Q - \pi_X\| - 1 \le \tfrac{1}{2}(\|I + 2Q - 2\pi_X\| - 1)$$
and $I + 2Q - 2\pi_X = (I - 2Q)(I - 2\pi_X)$. Thus $\|I + 2Q - 2\pi_X\| = \|I - 2Q\|$. Now $I - 2Q = (\phi^{**})^{-1}(I - 2\pi_Y)\phi^{**}$ so that $\|I - 2Q\| \le ab$. We thus obtain the estimate $\|qS\| \le \frac{1}{2}a(ab - 1)$ and the theorem will hold if $ab - 1 < 2a^{-1}$.

Now in case (1) we may assume that $a = 1$ and $b = \|I - 2\pi_Y\| \le \kappa_u(Y) < 3$. In case (2) we have $b = 1$ and then $a^2 - a - 2 = (a - 2)(a + 1) < 0$ or $a < 2$ is sufficient. ■

Remarks. The simplest case when this theorem applies is when X is c_0. As we observed in Section 5, it is unknown whether every isomorphic predual of ℓ_1 with property (u) is isomorphic to c_0. However, if Y is an isometric predual with $\kappa_u(Y) < 3$ or an isomorphic predual with $d(Y^*, \ell_1) < 2$ and $\kappa_u(Y) = 1$ then Y is isomorphic to c_0.

We conclude this section by showing that c_0 is the only isometric predual of ℓ_1 which is a u-ideal (for strict u-ideals see Theorem 5.7 and the following remarks).

PROPOSITION 7.8. *Let X be a real Banach space such that X^* is isometric to ℓ_1. If X is a u-ideal then X is isometric to c_0.*

Proof. Let $P : X^{***} \to X^{***}$ be a projection such that $\ker P = X^\perp$ and $\|I - 2P\| = 1$. We show that $P(X^{***})$ contains X^*, from which it follows that X is a strict u-ideal and the proof will be complete. Let π denote the canonical projection of X^{***} onto X^*. Let e be an extreme point of B_{X^*}.

Let $F = \{t \in B_{X^{***}} : \pi(t) = e\}$. If we let $S = 2P - I$ then it is clear that $S(F) = F$. Further if E is the set of extreme points of F then $S(E) = E$. Since E is contained in the set of extreme points of $B_{X^{***}}$ and X^{***} is isometric to ℓ_1^{**} it follows that E is a linearly independent set. If $f \in E$ then $Sf + f = 2Pf = 2Pe$. Thus if $f, g \in E$ we have $Sf - Sg = g - f$. Using linear independence we have $Sf = g$ and $Sg = f$. As this is true for any pair f, g we conclude that E contains at most two points. Now, from the fact that X is separable it is clear that F is a weak* G_δ-subset of $B_{X^{***}}$. Since F is either a point or a line segment we conclude that each $f \in E$ is a G_δ-point. Since X^* is weakly sequentially complete this means that $E \subset X^*$ and so

$F \subset X^*$; thus $F = \{e\}$ and so $Se = e$ and $Pe = e$. Since the norm-closed span of the extreme points coincides with X^* this concludes the proof. ∎

Remark. We do not know if every separable Lindenstrauss space which is a u-ideal is isometric to c_0. If X^* is isometric to $C[0,1]^*$ then the above argument shows at least that $V \cap X^*$ contains all discrete measures.

8. Spaces of operators. Let X be a separable Banach space. We shall say that a sequence of compact operators $K_n : X \to X$ is a *compact approximating sequence* if $\lim_{n\to\infty} K_n x = x$ for every $x \in X$. If each K_n is actually finite-rank then we will say that (K_n) is an *approximating sequence*.

Extending an idea introduced in [10] we will say that a Banach space X has *(UKAP)* if there is a compact approximating sequence $K_n : X \to X$ such that $\lim_{n\to\infty} \|I - 2K_n\| = 1$. If X is a complex Banach space we will say that X has *complex (UKAP)* if there is a compact approximating sequence such that $\lim_{n\to\infty} \|I - (1+\lambda)K_n\| = 1$ whenever $|\lambda| = 1$.

LEMMA 8.1. (1) *Let X be a separable Banach space. Then X has (UKAP) if and only if for every $\varepsilon > 0$ there is a sequence (A_n) of compact operators such that for every $x \in X$ and every n and every $\theta_j = \pm 1$, $1 \le j \le n$, we have $\sum_{n=1}^{\infty} A_n x = x$ and*

$$\Big\|\sum_{j=1}^{n} \theta_j A_j x\Big\| \le (1+\varepsilon)\|x\|.$$

(2) *Let X be a separable complex Banach space. Then X has complex (UKAP) if and only if for every $\varepsilon > 0$ there is a sequence (A_n) of compact operators such that for every $x \in X$ we have $\sum_{n=1}^{\infty} A_n x = x$ and for every n and every $\theta_1, \ldots, \theta_n$ with $|\theta_j| \le 1$ for $1 \le j \le n$,*

$$\Big\|\sum_{j=1}^{n} \theta_j A_j x\Big\| \le (1+\varepsilon)\|x\|.$$

Proof. We will prove only (2). (Compare [10] for the case when X has an approximating sequence of finite rank operators.) First assume X has complex (UKAP). Pick $\eta_m > 0$ to be a sequence such that $\prod(1+\eta_m) < 1+\varepsilon$. We may then choose a compact approximating sequence (K_n) such that $\|K_m K_n - K_n\| \le (4m)^{-1}\eta_m$ whenever $n < m$ and such that if $|\lambda| = 1$ then $\|I - (1+\lambda)K_m\| \le 1 + (4m)^{-1}\eta_m$. Let $K_0 = 0$ and then set $A_n = K_n - K_{n-1}$ for $n \ge 1$. We will show by induction that if $|\lambda_j| = 1$ for $1 \le j \le n$ then

$$\Big\|I - K_n + \sum_{j=1}^{n} \lambda_j A_j\Big\| \le \prod_{j=1}^{n} (1+\eta_j).$$

For $n = 1$ this is clear. Now suppose it is established for $n = m$. Suppose $|\lambda_j| = 1$ for $1 \le j \le m+1$. Let $\theta_j = \lambda_j/\lambda_{m+1}$ for $1 \le j \le m+1$. Then if $S = I - K_m + \sum_{j=1}^{m} \theta_j A_j$,

$$\|S\| \le \prod_{j=1}^{m} (1+\eta_j).$$

Thus

$$\|(I - (1-\lambda_{m+1})K_{m+1})S\| \le \Big(1 + \frac{\eta_{m+1}}{4m+4}\Big) \prod_{j=1}^{m} (1+\eta_j).$$

Now notice that $\|K_{m+1}A_j - A_j\| \le 2(4m+4)^{-1}\eta_{m+1}$ for $1 \le j \le m$. Thus

$$K_{m+1}S = K_{m+1} - K_m + \sum_{j=1}^{m} \theta_j A_j + V$$

where

$$\|V\| \le \frac{(2m+1)\eta_{m+1}}{4m+4}.$$

Thus

$$K_{m+1}S = \sum_{j=1}^{m+1} \theta_j A_j + V.$$

It follows that

$$(I - (1-\lambda_{m+1})K_{m+1})S = I - K_{m+1} + \lambda_{m+1} \sum_{j=1}^{m+1} \theta_j A_j - (1-\lambda_{m+1})V.$$

Combining, we have

$$\Big\|I - K_{m+1} + \sum_{j=1}^{m+1} \lambda_j A_j\Big\| \le \prod_{j=1}^{m+1} (1+\eta_j)$$

as required. This completes the induction. It now follows by convexity that if $|\theta_j| \le 1$ for $1 \le j \le n$ then for all $m \ge n$,

$$\Big\|I - K_m + \sum_{j=1}^{n} \theta_j A_j\Big\| \le 1 + \varepsilon.$$

Letting $m \to \infty$ yields that $\|\sum \theta_j A_j\| \le 1 + \varepsilon$ as required.

For the converse, notice that given ε we have a sequence (A_n) of compact operators such that if $K_n = \sum_{j=1}^{n} A_j$ then (K_n) is a compact approximating sequence and for $x \in X$ and $|\lambda| = 1$,

$$\|x - (1+\lambda)K_n x\| = \Big\|\sum_{j=1}^{n} \lambda A_j x - \sum_{j=n+1}^{\infty} A_j x\Big\| \le (1+\varepsilon)\|x\|. \ ∎$$

PROPOSITION 8.2. *Let X be a separable Banach space. If X has (UKAP) (resp. complex (UKAP)) then X is a u-ideal (resp. an h-ideal) and $\mathcal{K}(X)$ is a u-ideal (resp. an h-ideal) in $\mathcal{L}(X)$.*

Proof. The fact that X is a u-ideal (resp. an h-ideal) follows easily from Proposition 4.1. The remainder of the conclusion follows easily from Proposition 3.6. In fact, for the u-ideal version, if $\mathcal{F}$ is a finite-dimensional subspace of $\mathcal{K}(X)$ and $\varepsilon > 0$ we can find K compact such that $\|KS - S\| \le \varepsilon\|S\|$ for $S \in \mathcal{F}$ and $\|I - 2K\| \le 1 + \varepsilon$. Then consider $A(S) = KS$ for $S \in \mathcal{L}(X)$. ∎

Under certain circumstances we can prove the converse.

THEOREM 8.3. (a) *Let X be a separable reflexive Banach space. Then X has (UKAP) (resp. complex (UKAP)) if and only if $\mathcal{K}(X)$ is a u-ideal (resp. an h-ideal) in $\mathcal{L}(X)$.*

(b) *Let X be a complex Banach space such that X^* is separable. Then X has complex (UKAP) if and only if $\mathcal{K}(X)$ is an h-ideal in $\mathcal{L}(X)$ and X is an h-ideal.*

Proof. We consider (a) for the case of u-ideals. Denote by $\mathcal{P} : \mathcal{L}(X)^* \to \mathcal{L}(X)^*$ the projection with $\ker \mathcal{P} = \mathcal{K}(X)^\perp$ and by $\mathcal{T} : \mathcal{L}(X) \to \mathcal{K}(X)^{**}$ the induced operator as described in Section 2. For $x \in X$ and $x^* \in X^*$ let $x \otimes x^* \in \mathcal{K}(X)^*$ be the linear functional given by $\langle K, x \otimes x^* \rangle = \langle Kx, x^* \rangle$. This functional has a natural extension to $\mathcal{L}(X)$, also denoted by $x \otimes x^*$.

Now let $u \in X$ and $v^* \in X^*$ be points of Fréchet smoothness with $\|u\| = \|v^*\| = 1$. We suppose $u^* \in S_{X^*}$ and $v \in S_X$ satisfy $u^*(u) = v^*(v) = 1$. Let $A : X \to X$ be the rank-one operator given by $Ax = v^*(x)u$. A is a point of Fréchet smoothness in $\mathcal{L}(X)$. For real α we have $\|A + \alpha I\| = 1 + \alpha u^*(v) + o(|\alpha|)$. Hence in $\mathcal{K}(X)^{**}$ we obtain

$$\|A + \alpha \mathcal{T}(I)\| \le 1 + \alpha u^*(v) + o(|\alpha|).$$

In particular,

$$\langle v \otimes u^*, A + \alpha \mathcal{T}(I) \rangle \le 1 + \alpha u^*(v) + o(|\alpha|).$$

Hence

$$\langle v \otimes u^*, \mathcal{T}(I) \rangle = u^*(v).$$

Now since the points of Fréchet smoothness form a dense G_δ in both X and X^* we have for every $x \in X$, $x^* \in X^*$,

$$\langle x \otimes x^*, \mathcal{T}(I) \rangle = x^*(x).$$

Now by Lemma 2.2, there is a net (K_d) in $\mathcal{K}(X)$ such that K_d converges weak* to $\mathcal{T}(I)$ and $\limsup \|I - 2K_d\| = 1$. But then $K_d \to I$ for the weak operator topology. Hence for each d we can find $L_d \in \operatorname{co}\{K_e : e \ge d\}$ such that $L_d \to I$ for the strong operator topology. It follows that there is a compact approximating sequence (M_n) in $\mathcal{K}(X)$ such that $\lim \|I - 2M_n\| = 1$.

(b) In this case with X not necessarily reflexive, for $x^* \in X^*$ and $x^{**} \in X^{**}$ we use $x^* \otimes x^{**}$ to denote the element of $\mathcal{K}(X)^*$ given by $\langle S, x^* \otimes x^{**} \rangle = x^{**}(S^* x^*)$. The formula then defines $x^* \otimes x^{**} \in \mathcal{L}(X)^*$.

Proceeding as in part (a) we define $\mathcal{T} : \mathcal{L}(X) \to \mathcal{K}(X)^{**}$. It is then possible to define an operator $H : X^{**} \to X^{**}$ such that

$$\langle x^* \otimes x^{**}, \mathcal{T}(I) \rangle = \langle x^*, Hx^{**} \rangle.$$

We will argue that H is hermitian. In fact, suppose $|\lambda| = 1$. It follows from the fact that $\mathcal{P}$ is hermitian that $\|I - (1 + \lambda)H\| \le 1$. Thus if ϕ is a state on $\mathcal{L}(X^{**})$ then $|1 - (1 + \lambda)\phi(H)| \le 1$. Hence $\phi(H)$ is real and further $0 \le \phi(H) \le 1$. Hence H is hermitian.

Next we argue that there is an hermitian $H_0 : X \to X$ such that $H = H_0^{**}$. In fact, for any real t, $\exp(itH)$ is an isometric isomorphism on X^{**}. We recall that X is necessarily a strict h-ideal by Theorem 6.6. Thus we can apply Theorem 5.7 to deduce that $\exp(itH)$ maps X to X and is weak*-continuous. Differentiating we conclude that $H = H_0^{**}$ where $H_0 : X \to X$ is the restriction of H.

Next we recall that the collection of points of Fréchet smoothness in X forms a dense G_δ as do the points of Gateaux smoothness in X^*. Let us suppose that $u \in S_X$ is a point of Fréchet smoothness and that $u^* \in S_{X^*}$ is the exposed functional corresponding to u. Similarly suppose $v^* \in S_{X^*}$ is a point of Gateaux smoothness and v^{**} is the corresponding exposed functional in $S_{X^{**}}$. Define, as in (a), the rank-one operator $Ax = v^*(x)u$. We argue that

$$\|A + \zeta I\| = 1 + \Re\zeta(v^{**}(u^*)) + o(|\zeta|).$$

In fact,

$$\|A + \zeta I\| \ge \Re v^{**}(A^* u^* + \zeta u^*) = 1 + \Re(\zeta v^{**}(u^*)).$$

Conversely, for any ζ we may pick $x^*(\zeta) \in S_{X^*}$ so that $0 \le x^*(\zeta)(u) \le 1$ and

$$\|(A^* + \zeta I)(x^*(\zeta))\| \ge \|A + \zeta I\| - |\zeta|^2.$$

Letting $\zeta \to 0$ we observe that if x^* is any weak*-cluster point then $0 \le x^*(u) \le 1$ and $\|Ax^*\| = 1$. Hence $\lim_{\zeta \to 0} x^*(\zeta) = u^*$ weak*. However, this implies, since u is a point of Fréchet smoothness, that $\lim_{\zeta \to 0} \|x^*(\zeta) - u^*\| = 0$. It now follows immediately from the Gateaux smoothness of the norm at v^* that

$$\|A + \zeta I\| = 1 + \Re(\zeta v^{**}(u^*)) + o(|\zeta|).$$

Now as in case (a) we have

$$\|\mathcal{T}(A) + \zeta \mathcal{T}(I)\| \le 1 + \Re(\zeta v^{**}(u^*)) + o(|\zeta|)$$

and hence

$$\Re\langle (A^{**}+\zeta H)v^{**},u^*\rangle \le 1+\Re\langle \zeta v^{**},u^*\rangle + o(|z|)\,.$$

It follows that

$$\langle Hv^{**},u^*\rangle = \langle v^{**},u^*\rangle\,.$$

If we fix u^*, the collection of all v^{**} as v^* ranges over all points of Gateaux smoothness spans a weak*-dense subspace of X^{**}. Since $H=H_0^{**}$ is weak*-continuous it follows that

$$\langle Hx^{**},u^*\rangle = \langle x^{**},u^*\rangle$$

for all $x^{**}\in X^{**}$. Hence $H_0^*u^*=u^*$. But again the collection of such u^* spans a weak*-dense subspace of X^* and so $H_0=I$.

Now by Lemma 2.2 there is a net (K_d) in $\mathcal{K}(X)$ such that $K_d\to\mathcal{T}(I)$ weak* and $\limsup\|I-(1+\lambda)K_d\|=1$ whenever $|\lambda|=1$. The remainder of the argument is as in case (a) since we obviously have $K_d\to I$ in the weak operator topology. ∎

We now turn to the question of when, for a separable complex Banach space X, $\mathcal{K}(X)$ is an h-ideal (in $\mathcal{K}(X)^{**}$). If X is reflexive and has the compact approximation property then $\mathcal{K}(X)^{**}$ can be identified with $\mathcal{L}(X)$ and so the preceding theorem provides a complete answer: X must have complex (UKAP).

We will consider the situation when X^* is separable. In this case a compact approximating sequence (K_n) is called *shrinking* if (K_n^*) is also a compact approximating sequence for X^*.

LEMMA 8.4. *Let X be an h-ideal such that X^* is separable. If X has a compact approximating sequence (K_n) such that $\lim_{n\to\infty}\|K_n\|=1$ then X has a shrinking compact approximating sequence (L_n) with $\lim_{n\to\infty}\|L_n\|=1$.*

Proof. This is essentially proved in [31], where M-ideals are considered. We remark first that X is a strict h-ideal by Theorem 6.6. From Proposition 2.7, X^* has no proper norming subspace. Now let S be any cluster point of (K_n^{**}) in $\mathcal{L}(X^{**})$ for the weak*-operator topology; then $\|S\|=1$ and $Sx=x$ for $x\in X$. Hence ([31]) $S=I_{X^{**}}$; it follows now that $K_n^{**}\to I$ for the $\sigma(X^{**},X^*)$-operator topology. It follows that K_n^* converges to the identity for the weak operator topology on X^* and so some sequence of convex combinations L_n of K_n is shrinking. ∎

In general let us remark that if X^* is separable and X has the shrinking compact approximation property then $\mathcal{K}(X)^{**}$ can be canonically identified with $\mathcal{L}(X^*,X^{***})$. The identification is given by $\chi\to A$ where

$$\langle \chi, x^*\otimes x^{**}\rangle = \langle x^{**},Ax^*\rangle$$

and weak*-convergence (in $\mathcal{K}(X)^{**}$) coincides with the $\sigma(X^{***},X^{**})$-operator topology on bounded sets.

Under this identification each $S\in\mathcal{K}(X)$ is identified with the operator $S^*:X^*\to X^*\subset X^{***}$. $\mathcal{K}(X)$ is thus identified with the space of operators with range in X^* which are $\sigma(X^*,X)$-norm continuous on bounded sets.

THEOREM 8.5. *Let X be a separable complex Banach space with the metric compact approximation property and such that X^* is separable. Then:*

(a) *If X is an h-ideal, then there is an hermitian projection $\mathcal{T}:\mathcal{K}(X)^{**}\to Ba(\mathcal{K}(X))$.*

(b) *If X has complex (UKAP) then, in addition, $\kappa_{\rm h}(\mathcal{K}(X))=1$.*

Proof. (a) We first observe that X is a strict h-ideal (Theorem 6.6) and hence X^* is an h-summand. In particular, $X^*=Ba(X^*)$, i.e. X^* is weakly sequentially complete.

We now identify $Ba(\mathcal{K}(X))$ as a subspace of $\mathcal{L}(X^*,X^{***})$ using the identification described above. We claim that $Ba(\mathcal{K}(X))=\mathcal{L}(X^*)$. First suppose $L\in\mathcal{L}(X^*)$ and that (K_n) is a shrinking compact approximating sequence for X. Then $LK_n^*\in\mathcal{K}(X)$ and $LK_n^*\to A$ for the $\sigma(X^*,X^{**})$-operator topology. Conversely, suppose $S_n\in\mathcal{K}(X)$ and $S_n\to A$ for the $\sigma(X^*,X^{**})$-operator topology. Then if $x^*\in X^*$, $S_n^*x^*$ is weakly Cauchy in X^* and converges weak* in X^{***} to L^*x^*. Since X^* is weakly sequentially complete we have $L^*(X^*)\subset X^*$.

Now let $\pi:X^{***}\to X^*$ be the canonical projection. We define $\mathcal{T}(L)=\pi L$. Clearly since π is hermitian, so is $\mathcal{T}$. This completes the proof of (a).

(b) In this case, for any $\varepsilon>0$ there is a sequence (A_n) of compact operators such that $\sum_{n=1}^\infty A_nx=x$ for $x\in X$ and $\|\sum_{j=1}^n\theta_jA_j\|<1+\varepsilon$ whenever $|\theta_j|\le 1$ for $1\le j\le n$. Clearly since X^* is weakly sequentially complete we have $\sum_{n=1}^\infty A_n^*x^*=x^*$ for $x^*\in X^*$. Now suppose $L\in\mathcal{L}(X^*,X^*)$. Then $\sum_{n=1}^\infty LA_n^*=L$ for the $\sigma(X^*,X^{**})$-operator topology. It follows that $\kappa_{\rm h}(\mathcal{K}(X))=1$. ∎

In view of this theorem one might hope that under the hypothesis that X^* is separable and X has complex (UKAP) we might have that $\mathcal{K}(X)$ is an h-ideal. This, however, is false. In fact, we have the rather surprising result:

THEOREM 8.6. *Let X be a separable complex Banach space with the metric compact approximation property. Then the following conditions are equivalent:*

(1) *$\mathcal{K}(X)$ is an h-ideal (in $\mathcal{K}(X)^{**}$).*
(2) *X is an M-ideal (in X^{**}) and has complex (UKAP).*

Proof. (1)⇒(2). Observe first that both X and X^* are 1-complemented in $\mathcal{K}(X)$ so that both are h-ideals. Thus by Theorem 6.6, X is a strict h-ideal.

In particular, X^* is separable and X has a shrinking compact approximating sequence. Now identifying $\mathcal{K}(X)^{**}$ as explained above, we see that $I_{X^*} \in Ba(\mathcal{K}(X))$. Since $\kappa_h(\mathcal{K}(X)) = 1$ it follows quickly that X has (UKAP). We also note that the hermitian projection of $\mathcal{L}(X^*, X^{***})$ onto $Ba(\mathcal{K}(X))$ is unique and hence is given by $\mathcal{T}(L) = \pi L$ as in Theorem 8.5.

We now turn to showing that X is an M-ideal. Clearly it suffices to consider the case when X is nonreflexive. We first establish the following claim:

CLAIM 1. *Suppose $x \in X$, $x^{**} \in X^{**}$. Then there exists a net (x_d) in X such that $x_d \to x^{**}$ weak* and*

$$\limsup_d \|x + x^{**} - x_d\| \le \sup_{|\alpha|^2+|\beta|^2\le 1} \|\alpha x + \beta x^{**}\|.$$

Let us prove the claim. Suppose $\varepsilon > 0$. Pick any $\phi \in X^\perp$ with $\|\phi\| = 1$. It follows from Proposition 6.8 (or more general considerations) that since X^* is an h-summand there exists $f \in S_{X^*}$ such that

$$(1+\varepsilon)\|\alpha f + \beta\phi\| \ge |\alpha| + |\beta|$$

for all complex α, β. Pick then $\chi \in S_{X^{**}}$ such that $\chi(f) = \phi(\chi) = 1$ and $\|\chi\| < 1 + 2\varepsilon$.

Consider the operator $L : X^* \to X^{***}$ defined by

$$Lx^* = x^{**}(x^*)(f+\phi) + x^*(x)(\phi - f).$$

Then

$$\mathcal{T}(L)x^* = (x^{**}(x^*) - x^*(x))f.$$

Pick any sequence (y_n) in X such that $y_n \to x^{**}$ weak*. Then the operators $A_n \in \mathcal{K}(X)$ given by $A_n^* x^* = x^*(y_n - x)f$ converge weak* to $\mathcal{T}(L)$. Theorem 6.5 guarantees a sequence (B_n) in $\mathcal{K}(X)$ such that B_n converges weak* to $\mathcal{T}(L)$ and $\limsup \|L - 2B_n^*\| \le \|L\|$. Now by an application of Mazur's theorem we can find $S_n \in \mathrm{co}\{A_n, A_{n+1}, \ldots\}$ and $C_n \in \mathrm{co}\{B_n, B_{n+1}, \ldots\}$ such that $\lim \|S_n - C_n\| = 0$. Hence $\limsup \|L - 2S_n^*\| \le \|L\|$. Furthermore

$$S_n^* x^* = x^*(x_n - x)f$$

where x_n converges weak* to x^{**}.

Now we can write

$$(L - 2S_n^*)x^* = \langle x^*, x^{**} - x_n + x\rangle(f+\phi) + \langle x^*, x_n\rangle(\phi - f).$$

Thus

$$\langle \chi, (L - 2S_n^*)x^*\rangle = 2\langle x^*, x^{**} - x_n + x\rangle.$$

It follows that

$$2\|x^{**} - x_n + x\| \le (1+2\varepsilon)\|L - 2S_n^*\|.$$

Hence

$$\limsup_{n\to\infty} \|x + x^{**} - x_n\| \le \frac{1+2\varepsilon}{2}\|L\|.$$

We thus turn to estimating $\|L\|$. In fact, if $\xi^{**} \in B_{X^{**}}$ then $\xi^{**}(f) = a$ and $\phi(\xi^{**}) = b$ where $\max(|a|, |b|) \le 1$. Thus

$$\langle \xi^{**}, Lx^*\rangle = \langle x^*, (a+b)x^{**} + (b-a)x\rangle.$$

If we write $\alpha = (a+b)/2$ and $\beta = (b-a)/2$ then $|\alpha|^2 + |\beta|^2 \le 1$. Thus

$$\|L\| \le 2 \sup_{|\alpha|^2+|\beta|^2\le 1} \|\alpha x^{**} + \beta x\|.$$

We conclude that

$$\limsup_{n\to\infty} \|x + x^{**} - x_n\| \le (1+2\varepsilon) \sup_{|\alpha|^2+|\beta|^2\le 1} \|\alpha x + \beta x^{**}\|$$

and Claim 1 follows easily.

We now claim:

CLAIM 2. *For any $x^{**} \in X^{**}$ and $x \in X$ there is a net (x_d) in X such that x_d converges weak* to x^{**} and $\limsup_d \|x + x^{**} - x_d\| \le \max(\|x\|, \|x^{**}\|)$.*

In fact, for any such x, x^{**} let us define $F : \mathbb{C}^2 \to \mathbb{R}$ by

$$F(a,b) = \inf_{(x_d)} \limsup_d \|ax + b(x^{**} - x_d)\|$$

where (x_d) ranges over all nets in X converging weak* to x^{**}. We claim first that there is a single net (x_d) which suffices for all a, b. This is standard. Let $\varepsilon > 0$ and let $(a_j, b_j)_{j=1}^n$ be a finite set of points in $\mathbb{C}^2$. Let W be a convex weak*-open subset of X^{**}. Let $C_j = \{y \in W : \|a_j x + b_j(x^{**} - y)\| \le F(a_j, b_j) + \frac{1}{2}\varepsilon\}$. Consider the set $C = \{(y_j)_{j=1}^n \in X^n : y_j \in C_j\}$. We claim that $\inf_{y\in C, z\in X} \max_j(\|y_j - z\|) = 0$. Indeed, if not there exist $x_j^* \in X^*$ for $1 \le j \le n$ such that $\sum x_j^* = 0$ and $\inf_{y\in C} \Re \sum_{j=1}^n x_j^*(y_j) > 0$. But then $\sum_{j=1}^n x^{**}(x_j^*) \ne 0$ contrary to assumption. It now follows easily that we may pick $z \in W$ so that $\|a_j x + b_j(x^{**} - z)\| \le F(a_j, b_j) + \varepsilon$ for all j. Hence we can pick a fixed net (x_d) which works for all a, b. It now also follows that the lim sup becomes a limit and that F is a seminorm on $\mathbb{C}^2$.

Note that for fixed d there is net $y_e \to x^{**} - x_d$ weak* so that

$$\limsup_e \|ax + b(x^{**} - x_d - y_e)\| \le \max_{|\alpha|^2+|\beta|^2\le 1} \|\alpha a x + \beta b(x^{**} - x_d)\|.$$

Thus we can pick α_d, β_d with $|\alpha_d|^2 + |\beta_d|^2 \le 1$ so that

$$F(a,b) \le \|\alpha_d a x + \beta_d b(x^{**} - x_d)\|.$$

By passing to a subnet we can suppose that α_d, β_d converge to some α, β and hence since (x_d) is clearly bounded,

$$F(a,b) \le F(\alpha a, \beta b).$$

Iterating this condition we find a sequence (α_n,β_n) with $|\alpha_n|^2+|\beta_n|^2\le 1$ and $(a_n,b_n)_{n\ge 0}$ with $a_0=b_0=1$ so that $a_n=\alpha_n a_{n-1}$, $b_n=\beta_n b_{n-1}$ and $F(a_n,b_n)$ is increasing. Now $|a_n|,|b_n|$ are decreasing. If both converge to zero then we obtain $F(1,1)=0$. If say $|a_n|$ is bounded below then $|\alpha_n|$ converges to one; thus $\lim b_n=0$ and $F(1,1)=F(\lim|a_n|,0)\le\|x\|$. In the other case $F(1,1)\le\|x^{**}\|$. Claim 2 is then established.

Now by Proposition 4.3, this completes the argument for (1)⇒(2).

We now turn to (2)⇒(1). This will follow from Theorem 6.5 and Theorem 8.5 above. Notice first that since X is an M-ideal we know that X^* is separable and X has a shrinking compact approximating sequence (K_n) so that $\lim\|I-(1+\lambda)K_n\|=1$ whenever $|\lambda|=1$. Thus we identify $\mathcal{K}(X)^{**}$ with $\mathcal{L}(X^*,X^{***})$. Suppose $L\in\mathcal{L}(X^*,X^{***})$ and consider $\mathcal{T}(L)=\pi L\in\mathcal{L}(X^*,X^*)$. Then $K_n^*\pi L\to\pi L$ for the $\sigma(X^*,X^{**})$-operator topology. If $x^*\in B_{X^*}$ and $|\lambda|=1$ we have

$$\begin{aligned}\|Lx^*-(1+\lambda)K_n^*\pi Lx^*\| &= \|Lx^*-\pi Lx^*\|+\|(I-(1+\lambda)K_n^*)\pi Lx^*\|\\ &\le \|Lx^*-\pi Lx^*\|+\|I-(1+\lambda)K_n\|\|\pi Lx^*\|\\ &\le \|I-(1+\lambda)K_n\|\|Lx^*\|\,.\end{aligned}$$

Hence $\|L-(1+\lambda)K_n^*\pi L\|\le\|I-(1+\lambda)K_n\|\|L\|$ and the fact that $\mathcal{K}(X)$ is an h-ideal follows from Theorem 6.5. ∎

Remark. We are now ready to supply an example promised in Section 6 after Theorem 6.5. Let X be a strict h-ideal with complex (UKAP) which is not an M-ideal, e.g. $X=\mathbb{C}\oplus_1 c_0$ or $X=\ell_2(c_0)$. Then $Y=\mathcal{K}(X)$ is not an h-ideal. However, $\kappa_h(Y)=1$ and $Ba(Y)$ is complemented by an hermitian projection in Y^{**} from Theorem 8.5.

9. Commuting approximation properties.

In this section we make a few remarks concerning connections between some questions for u-ideals and a problem on commuting approximation properties. For convenience we restrict attention to the real case. In [10] it is shown that a separable Banach space with the metric approximation property (MAP) has the commuting (MAP) or (CMAP). In [53] it is shown that there is a separable reflexive space X with (CMAP) but failing to have a finite-dimensional decomposition (FDD); this space can be supposed to have the property that $\mathcal{K}(X)$ is an M-ideal in $\mathcal{L}(X)=\mathcal{K}(X)^{**}$, since it is a subspace of C_2 (see [10], [41]).

Recall that a separable Banach space X has (UMAP) if there is an approximating sequence (S_n) of finite-rank operators so that $\lim_{n\to\infty}\|I-2S_n\|=1$ (see [10]). We say that X has (UCMAP) if further X has a commuting approximating sequence (S_n) of finite-rank operators such that $\lim_{n\to\infty}\|I-2S_n\|=1$.

Of course, as observed in the previous section (Proposition 8.2) a space with (UKAP) must be a u-ideal. We start with a very simple proposition.

PROPOSITION 9.1. *Let X be a separable Banach space with (UKAP). Then:*

(1) *If X contains no copy of c_0 then X is a u-summand.*
(2) *If X contains no complemented copy of ℓ_1, then X is a strict u-ideal.*

Proof. (1) is Theorem 3.5. For (2) note that X^* cannot contain a copy of c_0. It follows easily that if $\varepsilon>0$ and we choose A_j as in Lemma 8.1 then $\sum_{j=1}^\infty A_j^*x^*=x^*$ for $x^*\in X^*$. But then if $x^{***}\in X^{***}$, $\pi(x^{***})=\sum_{j=1}^\infty A_j^{***}x^{***}$ and hence $\|I-2\pi\|\le 1+\varepsilon$. Hence $\|I-2\pi\|=1$. ∎

THEOREM 9.2. *Let X be a separable Banach space containing no complemented copy of ℓ_1. Then the following conditions on X are equivalent:*

(1) *X has (UMAP).*
(2) *X and X^* have (UMAP).*
(3) *X^* has (UMAP) and X is a strict u-ideal.*
(4) *X has (UCMAP).*
(5) *For every $\varepsilon>0$, X is isometric to a $(1+\varepsilon)$-complemented subspace of a Banach space Y with a shrinking $(1+\varepsilon)$-unconditional (FDD).*
(6) *X has (MAP) and for every $\varepsilon>0$ there is a Banach space Y with a 1-unconditional basis and a subspace Z such that $d(X,Z)<1+\varepsilon$.*

Proof. (1)⇔(2) is easy from the argument of the preceding proposition. Clearly (1)⇒(3) from Proposition 9.1. Also we have (4)⇒(1) immediately. Let us first complete the proof that the first four statements are equivalent.

(3)⇒(1). Let (S_n) be an unconditional approximating sequence for X^*. Then since X^* contains no copy of c_0 one may construct a projection $P: X^{***}\to X^*$ by $Px^{***}=\lim S_n^{**}x^{***}$. Clearly $\|I-2P\|=1$. However, if X is a strict u-ideal then $\|I-2\pi\|=1$ and so by Lemma 3.1, $P=\pi$.

Now consider the operators $S_n^*: X^{**}\to X^{**}$. Let $q: X^{**}\to X^{**}/X$ be the quotient map, and let $j_0: X\to X^{**}$ be the canonical embedding. Let $L_n=qS_n^{**}j_0: X\to X^{**}/X$. Then L_n^* maps $X^\perp$ to X^* and coincides with S_n^{**}. Thus L_n^* converges to zero for the strong operator topology, and this is enough to imply ([42]) that L_n converges to zero for the weak topology on $\mathcal{K}(X,X^{**}/X)$. It follows that we can find an approximating sequence (B_n) for X^* which is a sequence of convex combinations of S_n so that $\lim\|qB_n^*j_0\|=0$.

Now it follows from the Principle of Local Reflexivity (see [16, Lemma 2.9]) that for each n there is a projection Q_n of $\mathrm{span}(X\cup B_n^*(X))$ onto X with $\|Q_n\|<4$. Now it follows that $\lim\|(I-Q_n)B_n^*j_0\|=0$. Let $R_n=Q_nB_n^*j_0: X\to X$. Then $\limsup\|I-2R_n\|\le\limsup\|I-2B_n^*\|\le 1$. Further $\lim B_n^*x^{**}=x^{**}$ weak* for all $x^{**}\in X^{**}$ so it follows that

$\lim R_n x = x$ weakly for all $x \in X$. By a further convex combination argument we can then find a sequence of convex combinations T_n of R_n which is an approximating sequence with $\lim \|I - 2T_n\| = 1$.

(2)⇒(4). Here X has an approximating sequence (S_n) such that (S_n^*) is an approximating sequence for X^* and such that $\lim \|I - 2S_n\| = 1$. By Theorem 2 of [9] it also has a commuting approximating sequence (R_n) such that (R_n^*) is an approximating sequence for X^*. Then $R_n - S_n$ converges weakly to zero in $\mathcal{K}(X)$ and so there is an approximating sequence of convex combinations L_n of R_n such that $\lim \|I - 2L_n\| = 1$.

We now complete the proof by sketching the implications (1)⇒(5)⇒(6)⇒(1).

(1)⇒(5). Using the technique of [52] or [57] it is easy to show that X is $(1+\varepsilon)$-isomorphic to a 1-complemented subspace of a space with (FDD). The proof is completed by using an argument like that of [30] or [47] adapting the proof of [16, Theorem 3.3].

(5)⇒(6). By Theorem 1.g.5 of [48] any space with an unconditional (FDD) is isomorphic to a subspace of a space with unconditional basis, and the proof shows that the constants can be controlled [47]. The result then follows by interpolation (see [47]).

(6)⇒(1). First notice that if Y has a shrinking 1-unconditional basis, it is a strict u-ideal (see Section 4). Hence if Z is a subspace of Y, Z is also a strict u-ideal. Thus X is a strict u-ideal. Hence by Proposition 2.7 and [31, Proposition 2.5], X^* has (MAP); thus there is an approximating sequence (R_n) for X so that (R_n^*) is approximating for X^*.

Now let $J : X \to Y$ be an embedding with $\|J\|\|J^{-1}\| < 1+\varepsilon$ and let S_n be the partial sum operators for the basis of Y. Then $S_n J - JR_n$ converges to zero in $\mathcal{K}(X,Y)$. Thus by passing to a sequence of convex combinations B_n of S_n and T_n of R_n we may assume that $\lim \|B_n J - JT_n\| = 0$. Thus $\lim \|(I-2B_n)J - J(I-2T_n)\| = 0$ and hence $\limsup \|I - 2T_n\| < 1+\varepsilon$. As $\varepsilon > 0$ is arbitrary X has (UMAP). ∎

We remark that it is open whether (UMAP) implies (UCMAP) in general. For spaces not containing c_0 this can be reformulated:

COROLLARY 9.3. *Let Y be a separable Banach space not containing c_0 with (UMAP). Then Y has (UCMAP) if and only if Y is isometric to the dual of a strict u-ideal.*

Proof. If $Y = X^*$ where X is a strict u-ideal then by the preceding theorem X has (UCMAP) and it immediately follows that Y has (UCMAP). Conversely, suppose (S_n) is a commuting unconditional approximating sequence for Y. Let $X \subset Y^*$ be the closed linear span of $\bigcup S_n^*(Y^*)$. Then ([60]) (S_n^*) is an approximating sequence for X. We show that X is a predual of Y. In fact, if x^* is a linear functional on X there exists $\xi \in Y^{**}$ so that $\xi(x) = x^*(x)$ for $x \in X$ and $\|\xi\| = \|x^*\|$. But then $S_n^{**}\xi$ converges to some $y \in Y$ and clearly $\|y\| \le \|x^*\|$ but $x(y) = x^*(x)$ for $x \in X$. Thus X is an isometric predual of Y which has (UMAP) and hence is a strict u-ideal. ∎

Remark. Notice that in this corollary Y is a u-summand whenever it has (UMAP); Y has (UCMAP) if and only if $\ker P$ is weak*-closed where $P : Y^{**} \to Y$ is the associated projection. Note that L_1 (which fails (UMAP)) is a u-summand whose u-complement is not weak*-closed. In this context we also mention the example of Talagrand [62] of a separable Banach lattice Y not containing c_0 which is a dual of a Banach lattice, but not the dual of an order-continuous lattice. Y is then a u-summand but its u-complement is not weak*-closed. Similarly, an example is constructed in [28] of a separable dual space which is an L-summand without being the dual of an M-ideal.

10. Some open questions.

In this section we gather together some problems which are suggested by the current work.

QUESTION 1. If X^* is separable does there exist an equivalent norm on X so that any proper closed subspace of X^* has characteristic at most $1/2$?

Of course the answer here is positive for strict u-ideals; however, there are other cases when the answer is positive, e.g. in [32] it is shown that such a norm exists when X is quasi-reflexive of order one.

QUESTION 2. Let X be a separable Banach space such that for every closed subspace Z of X, every proper closed subspace M of Z^* has characteristic at most $1/2$. Is X a strict u-ideal?

Let us remark that Question 2 has an affirmative answer for the special case when $X = \mathcal{K}(Y)$ where Y is a separable reflexive Banach space with the approximation property. We indicate a brief proof. In this case $X^{**} = \mathcal{L}(Y)$. Let (ϕ_n) be a dense sequence in $\{I\}^\perp \subset \mathcal{K}(Y)^*$. We may pick R_n so that $\phi_k(R_n) = 0$ for $k \le n$, $\|R_n\| = 1$ and for some $\lambda_n \ge 0$, $\|\lambda_n I - 2R_n\| \le 1 + 1/n$. Clearly $\liminf \lambda_n \ge 1$. However, since $d(I, \mathcal{K}(Y)) = 1$ we clearly have $\limsup \lambda_n \le 1$ and hence $\lim \lambda_n = 1$. Note also that by passing to a subsequence we can suppose that $\lim R_n = \mu I$ weak* for some real μ. Now since the points of Fréchet smoothness are dense in $\mathcal{L}(Y)$ there exists an operator S which is a point of Fréchet smoothness. Then $\limsup \|S - 2SR_n\| \le \|S\|$ and SR_n converges weak* to μS. Arguing as in Theorem 5.5 we have $\mu = 1$. Thus $R_n \to I$ weak* and so Y has (UKAP) and thus $\mathcal{K}(Y) = X$ is a u-ideal.

QUESTION 3. If X has separable dual and property (u), can X be renormed to be a strict u-ideal (i.e. to have $\kappa_u(X) = 1$) or, in the complex case, a strict h-ideal?

QUESTION 4 ([28], Question IV.2). Let X have property (u) and be such that X^* is isomorphic to ℓ_1. Is X isomorphic to c_0?

See Theorem 6.7 above for partial results.

QUESTION 5. If X is a separable u-ideal containing no copy of ℓ_1, is X a strict u-ideal?

See Theorem 6.6 for the complex case (h-ideals).

QUESTION 6. If X is a separable Banach space containing no copy of c_0, does (UMAP) imply (UCMAP)?

By Corollary 9.3 this amounts to asking whether the u-complement of X in X^{**} is always weak*-closed.

QUESTION 7 ([31]). Let X be a subspace of c_0 with (AP). Does X have (MAP)?

By [31], X has (MAP) if it has the λ-(CBAP) for $\lambda < 2$. By the results of [10] this will be true if there is a space Y with (MAP) so that $d(X,Y) < 2$. Of course if X has (MAP) it actually has (UCMAP).

QUESTION 8. If X has (UCMAP) and Y is 1-complemented in X, does Y have (UCMAP)?

In the complex case, if X has a 1-unconditional basis then Y has a 1-unconditional basis (see [45], [55], [19]). The answer is also positive if X is a nondegenerate h-ideal containing no copy of c_0 (i.e. a nondegenerate h-summand). In fact, in this case X is the dual space of a strict h-ideal and by Theorem 6.9, Y is weak*-closed and thus also the dual of a strict h-ideal. Then we can quote Corollary 9.3.

QUESTION 9. Is it true that $\kappa_u(X) = 1$ if and only if X is a u-ideal in $Ba(X)$ (or $\kappa_h(X) = 1$ if and only if X is an h-ideal in $Ba(X)$)?

Related to Question 9 is:

QUESTION 10. Let X be a separable Banach space. Is there a norm-one projection Q on X^{***} such that $\ker Q = Ba(X)^{\perp}$? (In particular, what happens if $\kappa_u(X) = 1$?)

QUESTION 11. If X is a Banach space not containing c_0 and U is an invertible isometry on X^*, does it follow that U is weak*-continuous?

QUESTION 12. Is there a u-ideal X such that X^* is isometric to $C[0,1]^*$?

Added in proof (December 1992). Recently J. Alaminos has shown that Question 12 has a negative answer for separable X. He kindly allowed us to include his proof. By a theorem of Lazar and Lindenstrauss (Acta Math. 126 (1971), 165–193, Theorem 2.3) a separable Lindenstrauss space with nonseparable dual contains a 1-complemented copy of $C(\Delta)$ and hence also of the space c of convergent sequences. By Corollary 4.2 and Proposition 7.8 of this paper X cannot then be a u-ideal.

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