(Knowledge for Development)

KIBABII UNIVERSITY

UNIVERSITY EXAMINATIONS

2022/2023 ACADEMIC YEAR

THIRD YEAR FIRST SEMESTER

MAIN EXAMINATION

FOR THE DEGREE OF BACHELOR OF SCIENCE IN MATHEMATICS

COURSE CODE: **MAP 312**

COURSE TITLE: LINEAR ALGEBRA III

DATE: 13/12/2022 **TIME:** 9 AM -11 AM

INSTRUCTIONS TO CANDIDATES

Answer Question ONE and Any other TWO Questions

TIME: 2 Hours

This Paper Consists of 4 Printed Pages. Please Turn Over.

QUESTION 1. [30 MARKS]

a . Let $u = (2 + i, 0, 4 - 5i)$ and $v = (1 + i, 2 + i, 0)$.

(i). Determine the norms of the vectors

(ii). Find the distance between u and v.

[6 MARKS]

b . Compute the eigenvalues and the eigen vectors of the matrix $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

[6 MARKS]

c . State the Cayley-Hamilton Theorem. Verify the Cayley-Hamilton Theorem for the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ and hence find a^{-1}.

[6 MARKS]

d . Compute the matrix of the quadratic form $q(x); \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$q(x) = 2x_1 + (a + 4)x_2^2 + (a + 4)x_3^2 + 2ax_1x_2 + 6ax_1x_3.$$

Find the value of the real parameter a for which the quadratic form is positive definite.

[6 MARKS]

e . Which of the sets is orthogonal under the given inner product $\langle f; g \rangle = \int_0^\pi f(x)g(x)dx$ on $C[0; \pi]$?

(i). $\{1; \sin x\}$

(ii). $\{\cos x, \sin x\}$

[6 MARKS]

QUESTION 2. [20 MARKS]

a. Find the length and the inner product of the vectors $(\ 2 + 4i,\ \ -4i\ )$ and $(\ 2 - 4i,\ \ -4i\ )$

[9 MARKS]

b. Using the eigenvectors of $A = \begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix}$, compute the orthogonal matrix P and a diagonal matrix D such that $A = PDP^T$.

[11 MARKS]

QUESTION 3. [20 MARKS]

a. Compute the Jordan form and the Jordan basis for the matrix,

$$A = \begin{bmatrix} 4 & -7 & 5 \\ 1 & -3 & 4 \\ 1 & -6 & 7 \end{bmatrix}.$$

[10 MARKS]

b. Let V be an n-dimensional vector space over $\mathbb{C}$ and $T : V \to V$ a linear transformation. Recall that T is nilpotent, if $T^k = 0$, for some positive integer k.

(i). Prove that T is nilpotent, if all its eigenvalues are zero.

(ii). Conversely, prove that if T is nilpotent, then all the eigenvalues are zero.

[10 MARKS]

QUESTION 4. [20 MARKS].

a . Show that $S = \{(i, 0, 0), (i, i, 0), (0, 0, i)\}$ is a basis in $\mathbb{C}^3$.

[6 MARKS]

b . Let K be a field and suppose $A \in M_m(K)$ and $B \in M_n(K)$ have eigenvalues λ and μ in K. Show tha',

(i). $A \otimes I_n + I_m \otimes B$ has eigenvalue $\lambda + \mu$ and,

(ii). $A \otimes B$ has eigenvalue $\lambda\mu$.

[6 MARKS]

c . Let $A = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 3 \\ 1 & 0 \end{bmatrix}$. Compute the matrices $A \otimes I_n + I_m \otimes B$ and $A \otimes B$. Find thier eigenvalue.

[8 MARKS]

QUESTION 5. [20 MARKS].

a. Prove that for any quadratic form the underlying symmetric form is unique.

[6 MARKS]

b. Prove that the quadratic form

$$q(x) = 5x_1^2 + 8x_2^2 + 5x_3^2 - 4x_1x_2 + 8x_1x_3 + 4x_2x_3$$

is congruent to the quadratic form $q(xb_1 + xb_2 + xb_3) = 9x_2^2 + 9x_3^2$ where $\{b_1, b_2, b_3\}$ is the standard basis in $\mathbb{R}^3$.

[7 MARKS]

c. (i). Define Hermitian matrix.

(ii). Deduce if the following Hermitian matrix is positive definite.

$$M = \begin{bmatrix} 5 & i & 2-i \\ -i & 4 & 1-i \\ 2+i & 1+i & 3 \end{bmatrix}$$

[7 MARKS]