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Banach Ideals of Operators with Applications

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We present some results concerning the general theory of Banach ideals of operators and give several applications to Banach space theory. We give, in Section 3, new proofs of several recent results, as well as new operator characterizations of the $\mathscr{L}_p$-spaces of Lindenstrauss and Pelczynski. In Section 4 we prove that the space of absolutely summing operators from E to F is reflexive if both E and F are reflexive and E has the approximation property. Section 5 concerns Hilbert spaces. In particular, we compute the relative projection constant of Hilbert spaces in $L_p(\mu)$-spaces.

In this paper we present some results concerning the theory of Banach ideals of operators [31, 38]. Sections 1 and 2 are devoted to proving results, summarized in a table, concerning the ideal A and three associated ideals, A^*, A^Δ and A'. For Banach spaces with the metric approximation property it turns out that $A^*(E,F) = A^\Delta(E,F)$. It will become evident that in most cases A^Δ is the most important for the applications.

All properties we use concerning the approximation property, trace, and the notation and results concerning tensor products may be found in [12] and [13].

In Section 3 we give, in a relatively easy manner, new proofs of several recent results as well as new results concerning the $\mathscr{L}_p$-spaces of Lindenstrauss and Pelczynski [22]. The main result here is Theorem 3.6. We also give elementary proofs of results of Cohen [3] and Persson [27].

In Section 4 we prove that $\Pi_1(E,F)$, the space of absolutely summing operators from E to F, is reflexive if both E and F are

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reflexive and E has the approximation property. This result answers a question raised by P. Saphar at the conference on the geometry of normed linear spaces held in Jerusalem in June, 1972. Our applications in Section 5 concern Hilbert spaces. In particular, we compute the relative projection constant of Hilbert spaces in the $L_p(\mu)$-spaces. This answers a question of H. P. Rosenthal.

Although some of the ideas we use go back to the theory of tensor products as developed by Schatten [39] and Grothendieck [12, 13], our theory of conjugate ideals allows the proof of many results without the hypothesis of the (metric) approximation property [(m).a.p.] so critical to the latter work. In view of the remarkable example of Enflo [6], this is a real point of difference with the theory of tensor products. On the other hand we use tensor product methods in Section 4 and obtain, in the process, a rather curious fact concerning certain Banach ideals of operators.

1. Ideal Norms and Preliminaries

In the following $\mathscr{L}$ denotes the class of all bounded linear operators between arbitrary Banach spaces and $\mathscr{L}(E,F)$ the set of all such operators between specific Banach spaces E and F. Following Pietsch [31] we say that a class A of bounded linear operators is an *ideal* if for each set $A(E,F) = A \cap \mathscr{L}(E,F)$ one has

(a) if $x' \in E'$, $y \in F$ then $x' \otimes y \in A(E,F)$ ($x' \otimes y$ is the rank one operator given by $x' \otimes y(x) = \langle x, x' \rangle y$);

(b) $A(E,F)$ is a linear subset of $\mathscr{L}(E,F)$ for each E and F; and

(c) if $U \in \mathscr{L}(X, E)$, $T \in A(E,F)$, $V \in \mathscr{L}(F, Y)$, then $VTU \in A(X, Y)$.

The finite rank operators $\mathscr{F}$ obviously form the smallest ideal.

A function α on the operators T in an ideal A to the nonnegative real numbers is an *ideal norm* if one has

(d) if $x' \in E'$, $y \in F$ then $\alpha(x' \otimes y) = \| x' \| \| y \|$;

(e) if $S, T \in A(E,F)$ then $\alpha(S + T) \leqslant \alpha(S) + \alpha(T)$; and

(f) if $U \in \mathscr{L}(X, E)$, $T \in A(E,F)$ and $V \in \mathscr{L}(F, Y)$, then $\alpha(VTU) \leqslant \| V \| \alpha(T) \| U \|$.

We write $\alpha(T) < \infty$ iff $T \in A(E,F)$.

An ideal A with a norm α, $[A, \alpha]$, is a *Banach ideal* if each *component* $A(E,F)$ is a Banach space under α.

To any linear normed ideal $[A, \alpha]$ we associate three normed ideals as follows:

(i) *The dual ideal* $[A', \alpha']$: An operator T is in $A'(E, F)$ iff $T' \in A(F', E')$. Here $\alpha'(T) = \alpha(T')$. Our next ideal will be extremely useful in the applications.

(ii) The conjugate ideal $[A^{\Delta}, \alpha^{\Delta}]$: $A^{\Delta}(E, F)$ is the class of all operators $T \in \mathscr{L}(E, F)$ for which there is a $\rho > 0$ such that for any $L \in \mathscr{F}(F, E)$

$$|\operatorname{trace} LT| \leqslant \rho\alpha(L).$$

Here $\alpha^{\Delta}(T) = \inf \rho$, ρ satisfying the above inequality. It is clear that α^{Δ} is always a complete ideal norm.

(iii) *The adjoint ideal* $[A^*, \alpha^*]$: $A^*(E, F)$ is the class of all $T \in \mathscr{L}(E, F)$ for which there is a $\rho > 0$ such that for all finite dimensional Banach spaces X, Y and for all $V \in \mathscr{L}(X, E)$, $U \in A(Y, X)$ and $W \in \mathscr{L}(F, Y)$,

$$|\operatorname{trace} WTVU| \leqslant \rho \|W\| \|V\| \alpha(U).$$

The norm α^* is given by

$$\alpha^*(T) = \inf \rho,$$

ρ satisfying the above. Also α^* is always a complete ideal norm.

Following Pietsch [31] we say that an ideal $[A, \alpha]$ is *perfect* if $[A, \alpha] = [A^{**}, \alpha^{**}]$ with equality of the norms. We shall write "α is perfect" when no confusion is likely.

We begin with a result of Pietsch [31].

PROPOSITION 1.1. *Let* $T \in \mathscr{L}(E, F)$. *Then* $\alpha^*(T) \leqslant \alpha^{\Delta}(T)$, *and equality holds if both* E *and* F *have m.a.p.*

Proof. By the definitions $\alpha^*(T) \leqslant \alpha^{\Delta}(T)$. Assume now that E and F have m.a.p. Let $S = \sum_{i \leqslant n} y_i' \otimes x_i$ be a finite rank operator from F to E and let $\epsilon > 0$. Choose $U \in \mathscr{F}(E, E)$, $V \in \mathscr{F}(F, F)$ so that $\| U \| \leqslant 1 + \epsilon$, $\| V \| \leqslant 1 + \epsilon$, $Ux_i = x_i$ and $VTx_i = Tx_i$ for each $i = 1, 2, \ldots, n$ [16]. Let I be the inclusion of $V(F)$ into F, J the inclusion of $U(E)$ into E, and V_a, U_a the astrictions of V, U, respectively, to their ranges. Consider the sequence of maps

$$E \xrightarrow[T]{} F \xrightarrow[V_a]{} V(F) \xrightarrow[U_a SI]{} U(E) \xrightarrow[J]{} E.$$

Then

$$\begin{aligned} |\operatorname{trace}(ST)| &= \Big| \sum_{i\leqslant n} \langle Tx_i, y_i' \rangle \Big| \\ &= \Big| \sum_{i\leqslant n} \langle VTUx_i, y_i' \rangle \Big| \\ &= |\operatorname{trace}(SVTU)| \\ &= |\operatorname{trace}(USVT)| \\ &= |\operatorname{trace}(J(U_aSI)\, V_aT)| \\ &\leqslant \alpha^*(T)\, \alpha(U_aSI) \,\|J\|\, \|V_a\| \\ &\leqslant \alpha^*(T)\, \alpha(S)(1+\epsilon)^2. \end{aligned}$$

COROLLARY 1.2. $\alpha^{\Delta *} = \alpha^{**}$.

The following result was also first proved by Pietsch [31].

PROPOSITION 1.3. *For* $T \in \mathscr{L}(E, F)$,

$$\alpha^{**}(T) = \sup \alpha(UTV),$$

where the suprenum is taken over all finite dimensional X, Y *and all* $V \in \mathscr{L}(X, E)$, $U \in \mathscr{L}(F, Y)$ *with* $\| V \| = 1$, $\| U \| = 1$.

Proof. Given $\epsilon > 0$, there are finite dimensional spaces X, Y and operators $V \in \mathscr{L}(X, E)$, $U \in \mathscr{L}(F, Y)$ and $S \in A(Y, X)$ such that $\alpha^*(S) = \| U \| = \| V \| = 1$ and $\alpha^{**}(T) - \epsilon < |\operatorname{trace}(SUTV)|$. By Proposition 1.1,

$$\begin{aligned} |\operatorname{trace}(SUTV)| &\leqslant \alpha^{\Delta}(S)\, \alpha(UTV) \\ &= \alpha(UTV). \end{aligned}$$

Conversely, given any finite dimensional spaces X and Y, it is known (cf. [31, Theorem 4]) and easy to prove that $[A(X, Y), \alpha]'$ is naturally isometric to $[A^{\Delta}(Y, X), \alpha^{\Delta}]$, where $\langle W, S \rangle = \operatorname{trace}(SW)$. Let $V \in \mathscr{L}(X, E)$ and $U \in \mathscr{L}(F, Y)$ both have norm one. Choose $S \in A^{\Delta}(Y, X)$, $\alpha^{\Delta}(S) = 1$, so that $\alpha(UTV) = \operatorname{trace}(SUTV)$. Then

$$\begin{aligned} \alpha(UTV) &\leqslant \alpha^*(S)\, \alpha^{*\Delta}(UTV) \\ &= \alpha^{\Delta}(S)\, \alpha^{**}(UTV) \\ &\leqslant \alpha^{**}(T) \end{aligned}$$

by Proposition 1.1.

COROLLARY 1.4. *The norm α is perfect if and only if, for all T,*

$$\alpha(T) = \sup \alpha(UTV),$$

where the supremum is the same as in the preceeding proposition.

Remark. It follows easily from the corollary that α^* is perfect for any ideal norm α.

Recall the following result of Pietsch.

PROPOSITION 1.5 ([31]). *For any ideal norm α, $\alpha^{*\prime} = \alpha^{\prime *}$.*

PROPOSITION 1.6. *For any ideal norm α, $\alpha''^{\Delta'} \leqslant \alpha'^{\Delta} \leqslant \alpha^{\Delta'}$.*

Proof. Suppose $\alpha^{\Delta'}(T) < \infty$, where $T \in \mathscr{L}(E, F)$. Let $S \in \mathscr{F}(F, E)$. Then

$$\begin{aligned} |\operatorname{trace}(ST)| &= |\operatorname{trace}(T'S')| \\ &\leqslant \alpha^{\Delta}(T')\,\alpha(S') \\ &= \alpha^{\Delta'}(T)\,\alpha'(S). \end{aligned}$$

Hence, $\alpha'^{\Delta}(T) \leqslant \alpha^{\Delta'}(T)$.

For the other inequality, let $T \in \mathscr{L}(E, F)$ satisfy $\alpha'^{\Delta}(T) < \infty$, $\epsilon > 0$, and $S = \sum_{i \leqslant n} x_i'' \otimes y_i'$ be a finite rank operator from E' to F'. By [1] there is a mapping P: $[x_i''] \to E$ such that

$$\left| \sum_{i \leqslant n} \langle x_i'' - Px_i'', T'y_i' \rangle \right| < \epsilon,$$

and $\| P \| \leqslant 1$. Let $L = S' \mid F$. Then

$$\begin{aligned} |\operatorname{trace}(T'S)| &= |\operatorname{trace}(LT)| \\ &\leqslant |\operatorname{trace}(LTP)| + |\operatorname{trace}(LTP) - \operatorname{trace}(LT)| \\ &\leqslant |\operatorname{trace}(TPL)| + \epsilon \\ &\leqslant \alpha'^{\Delta}(T)\,\alpha'(PL) + \epsilon \\ &\leqslant \alpha'^{\Delta}(T)\,\| P \|\,\alpha'(S' \mid F) + \epsilon \\ &\leqslant \alpha'^{\Delta}(T)\,\alpha'(S') + \epsilon \\ &= \alpha'^{\Delta}(T)\,\alpha''(S) + \epsilon. \end{aligned}$$

Thus, $\alpha''^{\Delta'}(T) = \alpha''^{\Delta}(T') \leqslant \alpha'^{\Delta}(T)$.

COROLLARY 1.7. *For every* $T \in \mathscr{L}(E, F)$, $\alpha^{\Delta'}(T) = \alpha'^{\Delta}(T)$ *whenever* α *is perfect or both* E *and* F *are reflexive.*

Proof. Observe that $\alpha''^{*} = \alpha^{*}$ always holds, and that $\alpha^{**''} = \alpha''^{**} = \alpha^{**}$ by Proposition 1.5. Now if α is perfect, $\alpha = \alpha^{**}$, and, hence, $\alpha''^{\Delta'} = \alpha^{**''\Delta'} = \alpha^{**\Delta'} = \alpha^{\Delta'}$, and the result follows.

If E and F are reflexive, then $\alpha''(S) = \alpha(S'') = \alpha(S)$ for all $S \in \mathscr{L}(E', F')$. Therefore, $\alpha''^{\Delta}(T') = \alpha^{\Delta}(T')$ for all $T \in \mathscr{L}(E, F)$, that is, $\alpha''^{\Delta'}(T) = \alpha^{\Delta'}(T)$.

We end this section with some more results which will be useful in the applications.

PROPOSITION 1.8. *For all finite rank* L, $\alpha^{\Delta\Delta\Delta}(L) = \alpha^{\Delta}(L)$, *and* $\alpha^{\Delta\Delta}(L) \leqslant \alpha(L)$ *with equality when* α *is perfect.*

Proof. Let $L \in \mathscr{F}(E, F)$ and $U \in \mathscr{F}(F, E)$. Then $|\operatorname{trace}(UL)| \leqslant \alpha(U)\,\alpha^{\Delta}(L)$, therefore, $\alpha^{\Delta\Delta}(U) \leqslant \alpha(U)$ and also $\alpha^{\Delta\Delta\Delta}(L) \geqslant \alpha^{\Delta}(L)$. But also $|\operatorname{trace}(UL)| \leqslant \alpha^{\Delta\Delta}(U)\,\alpha^{\Delta}(L)$, so $\alpha^{\Delta\Delta\Delta}(L) \leqslant \alpha^{\Delta}(L)$, which implies equality. When α is perfect, $\alpha(L) \geqslant \alpha^{\Delta\Delta}(L) \geqslant \alpha^{\Delta\Delta**}(L) = \alpha^{****}(L)$ (by Corollary 1.2) $= \alpha(L)$.

COROLLARY 1.9. *For any* T, $\alpha^{\Delta}(T) \leqslant \alpha^{\Delta\Delta\Delta}(T)$ *with equality when* α *is perfect or* T *finite rank.*

COROLLARY 1.10. $\alpha^{\Delta\Delta\Delta\Delta} = \alpha^{\Delta\Delta}$.

COROLLARY 1.11. $\alpha^{**} \leqslant \alpha^{\Delta\Delta} \leqslant \alpha^{*\Delta}$.

Proof. $\alpha^{**} = \alpha^{\Delta\Delta**} \leqslant \alpha^{\Delta\Delta} \leqslant \alpha^{*\Delta}$ by Proposition 1.1, Corollary 1.2, and the remark after Corollary 1.4.

2. EXAMPLES

We now give some examples and a table showing the relationships between the various ideals. Proofs of the results listed in the tables will be given in Section 3.

(1) Let $C(E, F)$ denote the closure of $\mathscr{F}(E, F)$ in $\mathscr{L}(E, F)$ and $K(E, F)$ denote the compact operators from E to F. Then $[\mathscr{L}, \|\cdot\|]$, $[K, \|\cdot\|]$ and $[C, \|\cdot\|]$ are Banach ideals. If one orders the Banach ideals by $[A, \alpha] < [B, \beta]$ if and only if $A(E, F) \subset B(E, F)$ and $\alpha(T) \geqslant \beta(T)$ for $T \in A(E, F)$ then $[\mathscr{L}, \|\cdot\|]$ is the largest Banach ideal.

For a finite or denumerable set $\{x_1, ..., x_N\}$ in a Banach space E, let $\epsilon_p(\{x_i\}) = \sup\{(\sum_{i \leqslant N} |\langle x_i, f\rangle|^p)^{1/p} : \|f\| \leqslant 1\}$, if $1 \leqslant p < \infty$ and $\epsilon_\infty(\{x_i\}) = \sup\{\sup_{i \leqslant N} |\langle x_i, f\rangle| : \|f\| \leqslant 1\}$; $\alpha_p(\{x_i\}) = (\sum_{i \leqslant N} \|x_i\|^p)^{1/p}$, if $1 \leqslant p < +\infty$ and $\alpha_\infty(\{x_i\}) = \sup_{i \leqslant N} \|x_i\|$; and, $\sigma_p(\{x_i\}) = \sup\{|\sum_{i \leqslant N} \langle x_i, f_i\rangle| : \epsilon_{p'}(\{f_i\}) \leqslant 1\}$, $1/p + 1/p' = 1$.

We now continue with our list of Banach ideals.

(2) Let $[\Pi_p, \pi_p]$ denote the ideal of *p-absolutely summing* operators: $T \in \Pi_p(E, F)$ if there is a $\rho > 0$ such that $\alpha_p(\{Tx_i\}) \leqslant \rho\epsilon_p(\{x_i\})$ for all finite sets $\{x_1, ..., x_N\}$ in E. The norm π_p is given by $\pi_p(T) = \inf \rho$, ρ as above.

(3) Let $[D_p, d_p]$ denote the ideal of *strongly p-summing* operators: $T \in D_p(E, F)$ if there is a $\rho > 0$ such that $\sigma_p(\{Tx_i\}) \leqslant \rho\alpha_p(\{x_i\})$ for all finite sets $\{x_1, ..., x_N\}$ in E. Here $d_p(T) = \inf \rho$.

(4) Let $[J_p, j_p]$ denote the ideal of *Cohen p-nuclear* operators: $T \in J_p(E, F)$ if there is a $\rho > 0$ such that $\sigma_p(\{Tx_i\}) \leqslant \rho\epsilon_p(\{x_i\})$ for all finite sets $\{x_1, ..., x_N\}$ in E. Here $j_p(T) = \inf \rho$.

(5) Let $[I_p, i_p]$ denote the ideal of *p-integral* operators: $T \in I_p(E, F)$ if there is a probability measure space (Ω, μ) and operators $V \in \mathscr{L}(E, L_\infty(\mu))$ and $W \in \mathscr{L}(L_p(\mu), F'')$ such that $WjV = iT$, where j is the canonical injection of $L_\infty(\mu)$ into $L_p(\mu)$ and i the canonical injection of F into F''. The norm i_p is given by $i_p(T) = \inf \|V\| \|W\|$, the infimum taken over all probability measure spaces (Ω, μ) and operators V and W.

(6) Let $[N_p, \nu_p]$ denote the ideal of *p-nuclear* operators: $T \in N_p(E, F)$ if T has a representation $T = \sum_{i=1}^\infty f_i \otimes y_i$, where $f_i \in E'$, $y_i \in F$ and $\alpha_p(\{f_i\}) < +\infty$ and $\epsilon_{p'}(\{y_i\}) < +\infty$, $1/p + 1/p' = 1$. Here $\nu_p(T) = \inf \alpha_p(\{f_i\}) \epsilon_{p'}(\{y_i\})$, where the infimum is taken over all such representations of T. If $p = \infty$ there is the additional requirement that $f_i \to 0$ as $i \to \infty$.

(7) The ideal $[N^p, \nu^p]$ is defined as above, interchanging the roles of $\{f_i\}$ and $\{y_i\}$.

(8) Let $[N_p{}^Q, \nu_p{}^Q]$ denote the ideal of *quasi-p-nuclear* operators: $T \in N_p{}^Q(E, F)$ if there exists $\{f_i\}$ in E', $\alpha_p(\{f_i\}) < +\infty$ such that $\|Tx\| \leqslant (\sum_{i=1}^\infty |f_i(x)|^p)^{1/p}$ when $1 \leqslant p < \infty$, and $\|Tx\| \leqslant \sup_i |f_i(x)|$ when $p = \infty$ in which case it is also required that $f_i \to 0$ as $i \to \infty$. Here $\nu_p{}^Q(T) = \inf \alpha_p(\{f_i\})$, where the infimum is over all such sequences $\{f_i\}$.

(9) Let $[C_p, c_p]$ denote the ideal of operators *factoring compactly through l_p*: $T \in C_p(E, F)$ if there are $A \in C(E, l_p)$, $B \in C(l_p, F)$

such that $T = BA$. The norm c_p is given by $c_p(T) = \inf \| A \| \| B \|$, the infimum over all such factorizations $T = BA$.

(10) Let $[\Gamma_p, \gamma_p]$ denote the ideal of operators *factoring through* L_p: $T \in \Gamma_p(E, F)$ if for some $L_p(\mu)$, μ a positive measure, there are operators $A \in \mathscr{L}(E, L_p(\mu))$, $B \in \mathscr{L}(L_p(\mu), F'')$ such that $iT = BA$, where i is the canonical injection of F into F''. Here $\gamma_p(T) = \inf \| A \| \| B \|$, A, B as above.

The ideals $[\Pi_p, \pi_p]$ were studied by Grothendieck [12], Pietsch [29], Pietsch–Persson [28], Persson [27], and Saphar [36]. The ideals J_p and D_p were introduced by Cohen [3]. The ideals $[I_p, i_p]$, $[N_p, \nu_p]$, $[N^p, \nu^p]$, and $[N_p{}^Q, \nu_p{}^Q]$ were studied by Grothendieck [12], Pietsch–Persson [28], and Saphar [35].

The ideal $[C_p, c_p]$ was introduced by Johnson [15] and further studied by Figiel [7]. Finally the ideal $[\Gamma_p, \gamma_p]$ was introduced and studied by Kwapien [18].

It turns out that modifications of the ideals Γ_p, J_p, and I_p are useful for applications.

(11) Let $[I_{pq}, i_{pq}]$, $\infty \geqslant p \geqslant q \geqslant 1$, denote the ideal of operators *factoring through a diagonal* $B \in \mathscr{L}(L_p(\mu), L_q(\mu))$: $T \in I_{pq}(E, F)$ if for some positive measure μ there are operators $A \in \mathscr{L}(E, L_p(\mu))$, $B \in \mathscr{L}(L_p(\mu), L_q(\mu))$ and $C \in \mathscr{L}(L_q(\mu), F'')$, where B is of the form $Bf = f \cdot g$ for some fixed $g \in L_r(\mu)$ where $1/r = 1/q - 1/p$, such that $iT = CBA$ where i is the canonical injection of F into F''. Here $i_{pq}(T) = \inf \| A \| \| B \| \| C \|$.

We note that $I_{\infty q}(E, F) = I_q(E, F)$ with equality of norms.

(12) Let $[J_{pq}, j_{pq}]$, $\infty \geqslant p \geqslant q \geqslant 1$ denote the ideal of operators *factoring through* $D_q \circ \Pi_p$: $T \in J_{pq}(E, F)$ if iT admits a factorization as follows:

$$\begin{tikzcd} E \arrow[r, "T"] \arrow[dr, "U"'] & F \arrow[r, "i"] & F'' \\ & G \arrow[ur, "V"'] & \end{tikzcd}$$

where $U \in \Pi_p(E, G)$ and $V \in D_q(G, F'')$. Here $j_{pq}(T) = \inf \pi_p(U)\, d_q(V)$ the infimum taken over all U, V, G. Finally

(13) Let $[\Gamma_{pq}, \gamma_{pq}]$ be defined as in $[I_{pq}, i_{pq}]$, the only difference being that B ranges over all operators in $\mathscr{L}(L_p(\mu), L_q(\mu))$.

We now give Table I summarizing the relationships between the ideals above. In all cases in our table $1/p + 1/p' = 1$. The table is arranged so that when a property, such as m.a.p., is placed in a column, we assume this property for all entries which follow.

TABLE I

Ideal	Adjoint (*)	Conjugate (Δ)	Dual (′)
$[\mathscr{L}, \|\cdot\|]$	$[I_1, i_1]$	$[I_1, i_1]$	$[\mathscr{L}, \|\cdot\|]$
$[K, \|\cdot\|]$	$[I_1, i_1]$	$[I_1, i_1]$	$[K, \|\cdot\|]$
$[C, \|\cdot\|]$	$[I_1, i_1]$	$[I_1, i_1]$	$[C, \|\cdot\|]$
$[\Pi_p, \pi_p]$	$[I_{p'}, i_{p'}]$	$[I_{p'}, i_{p'}]$	$[D_{p'}, d_{p'}]$
$[D_p, d_p]$	$[I_{p'1}, i_{p'1}]$	$[I_{p'1}, i_{p'1}]$	$[\Pi_{p'}, \pi_{p'}]$
$[J_p, j_p]$	$[\Gamma_{p'}, \gamma_{p'}]$	$[\Gamma_{p'}, \gamma_{p'}]$	$[J_{p'}, j_{p'}]$
$[N_p{}^Q, \nu_p{}^Q]$	$[I_{p'}, i_{p'}]$	$[I_{p'}, i_{p'}]$ E' or F *has m.a.p.*	[a]
$[I_p, i_p]$	$[\Pi_{p'}, \pi_{p'}]$	$[\Pi_{p'}, \pi_{p'}]$	$[I_{p'1}, i_{p'1}]$
$[\Gamma_p, \gamma_p]$	$[J_{p'}, j_{p'}]$	$[J_{p'}, j_{p'}]$	$[\Gamma_{p'}, \gamma_{p'}]$
$[I_{pq}, i_{pq}]$	$[J_{q'p'}, j_{q'p'}]$	$[J_{q'p'}, j_{q'p'}]$	$[I_{q'p'}, i_{q'p'}]$
$[J_{pq}, j_{pq}]$	$[I_{q'p'}, i_{q'p'}]$[b]	$[I^{*\Delta}_{q'p'}, i^{*\Delta}_{q'p'}]$	$[J_{q'p'}, j_{q'p'}]$ E' or F'' *has m.a.p.*
$[N_p, \nu_p]$	$[\Pi_{p'}, \pi_{p'}]$	$[\Pi_{p'}, \pi_{p'}]$	$[N^p, \nu^p]$
$[N^p, \nu^p]$	$[D_p, d_p]$	$[D_p, d_p]$	$[N_p, \nu_p]$
$[C_p, c_p]$	$[J_{p'}, j_{p'}]$	$[J_{p'}, j_{p'}]$	$[C_{p'}, c_{p'}]$

[a] This entry is the closure of $\mathscr{F}(E, F)$ in the norm d_p.

[b] A. Pietsch has informed us that J. T. Lapreste has shown that I_{pq} is a perfect ideal. Thus, I_{pq}^{**} may be replaced by I_{pq} throughout this paper.

[1] See the remark following Table I.

We remark that we prove some results in the table with weaker hypotheses, but to list these results separately would make the table too cumbersome.

Observe that from the table it follows that if A is any perfect ideal then $I_1 \subset A \subset \mathscr{L}$. Also from the table Γ_p, Π_p, J_{pq} and their adjoints and duals are perfect. In particular $I_1 \subset J_p$ for all p; this result of Cohen [3] will be used later. We point out that although the m.a.p. hypothesis is not required to prove results concerning the adjoint ideals, the conjugate ideals are naturally more useful to consider in the applications which follow. We, therefore, prove the results on the conjugate ideals under the weakest possible hypotheses.

It will be helpful to the reader to refer to Table I occasionally. We will first briefly discuss what is known about the adjoint and dual ideals in the well known cases, and then describe the conjugate ideals.

It is known [31] that $[I_p{}^*, i_p{}^*] = [\Pi_{p'}, \pi_{p'}]$ with equality of norms and that $[I_p, i_p]$ is perfect. Also, $[\Pi_p{}', \pi_p{}'] = [D_{p'}, d_{p'}]$ [3], and the latter ideal is perfect since $\alpha'^* = \alpha^{*\prime}$ for every ideal norm α, and since α^* is always perfect. The equality $[\Gamma_p{}^*, \gamma_p{}^*] = [J_{p'}, j_{p'}]$

is contained in Grothendieck [12, 13] for $p = 1, 2, \infty$, and is given by Kwapien [18] for arbitrary p, $1 \leqslant p \leqslant \infty$. No detailed proof that Γ_p is perfect is given in [18]; we shall later give a proof of this fact in a more general case. It is immediate from the factorization definition that $\gamma_p{}' = \gamma_{p'}$, and the formula $j_p{}' = j_{p'}$ again follows since $\alpha'^* = \alpha^{*\prime}$. It is well known that $[K', \|\cdot\|] = [K, \|\cdot\|]$ (cf. [5]), and by the local reflexivity principle [16] $[C', \|\cdot\|'] = [C, \|\cdot\|]$.

We begin by recalling two well known results from Grothendieck (cf. [12, Definition 7, p. 126, Proposition 35(B) and Proposition 39(B_1)]).

Lemma 2.1. *A map $T \in \mathscr{L}(E, F)$ satisfies $i_1(T) \leqslant \rho$ iff $|\operatorname{trace}(ST)| \leqslant \rho \| S \|$ for every $S \in \mathscr{F}(F, E)$; that is, $i_1 = \| \ \|^{\Delta}$.*

Lemma 2.2. *For $T \in \mathscr{F}(E, E)$,*

(a) *$|\operatorname{trace}(T)| \leqslant \nu_1(T)$ whenever E has a.p.;*

(b) *$\nu_1(T) = i_1(T)$ whenever E has m.a.p.*

Lemma 2.3. *If α is any ideal norm, $T \in \mathscr{L}(E, F)$ and $S \in \mathscr{L}(F, G)$, then $i_1(ST) \leqslant \alpha^{\Delta}(T)\, \alpha(S)$ and $i_1(ST) \leqslant \alpha(T)\, \alpha^{\Delta}(S)$.*

Proof. Let $L \in \mathscr{F}(G, E)$. Then $|\operatorname{trace}(T(LS))| \leqslant \alpha(LS)\, \alpha^{\Delta}(T) \leqslant \alpha(S)\, \alpha^{\Delta}(T) \| L \|$. Since $i_1 = \| \ \|^{\Delta}$, the first inequality follows. The other proof is the same.

With assumptions involving m.a.p., the inequality of Proposition 1.8 may be improved.

Proposition 2.4. *Let $T \in \mathscr{L}(E, F)$, where one of E, F has m.a.p. Then $\alpha^{\Delta\Delta}(T) \leqslant \alpha(T)$, with equality when α is perfect.*

Proof. Suppose E has m.a.p. and let $S \in \mathscr{F}(F, E)$. Then ST is nuclear and by Lemma 2.2(a), $|\operatorname{trace}(ST)| \leqslant \nu_1(ST)$. But by Lemma 2.2(b), $\nu_1(ST) = i_1(ST)$, so that $|\operatorname{trace}(ST)| \leqslant \alpha(T)\, \alpha^{\Delta}(S)$ by Lemma 2.3. This gives $\alpha^{\Delta\Delta}(T) \leqslant \alpha(T)$. The equality when α is perfect follows from Corollary 1.11.

Theorem 2.5. *Let $T \in \mathscr{L}(E, F)$ and $1/p + 1/p' = 1$.*

(a) $\pi_p{}^{\Delta} = i_{p'}$.

(b) *$i_{p'}^{\Delta}(T) \geqslant \nu_{p'}^{\Delta}(T) \geqslant \pi_p(T)$, with equality if T is finite rank, or if either E or F has m.a.p.*

(c) $j_p{}^{\Delta} = \gamma_{p'}$.

(d) *$\gamma_{p'}^{\Delta}(T) \geqslant c_{p'}^{\Delta}(T) \geqslant j_p(T)$, with equality if T is finite rank, or if either E or F has m.a.p.*

Proof of (a). For one inequality we have that $i_{p'} = \pi_p{}^* \leqslant \pi_p{}^\Delta$. For the other inequality let $T \in I_{p'}(E, F)$, $\epsilon > 0$ and i_F be the canonical embedding of F into F''. If $p < \infty$, let $L \in \mathscr{F}(F, E)$ and choose K, D, R so that $\| R \| = \| K \| = 1$, $i_{p'}(D) < i_{p'}(T) + \epsilon$ and so that the following diagram commutes:

$$\begin{tikzcd} F \arrow[r, "L"] & E \arrow[r, "T"] & F \arrow[r, "i_F"] & F'' \\ & L_\infty(\mu) \arrow[rr, "D"] \arrow[from=ul, phantom] & & L_{p'}(\mu) \arrow[u, "R"'] \end{tikzcd}$$

$$F \xrightarrow{L} E \xrightarrow{T} F \xrightarrow{i_F} F'', \quad E \xrightarrow{K} L_\infty(\mu) \xrightarrow{D} L_{p'}(\mu) \xrightarrow{R} F''.$$

Then, $\operatorname{trace}(TL) = \operatorname{trace}(i_F TL) = \operatorname{trace}(RDKL) = \operatorname{trace}((KL)'' RD)$, since L is finite rank. Since $L_\infty(\mu)$ has m.a.p., by Lemma 2.2

$$|\operatorname{trace}(KL)''RD)| \leqslant i_1((KL)''RD) \leqslant \pi_p((KL)'')\, i_{p'}(RD).$$

Since $\alpha(T'') = \alpha(T)$ whenever α is perfect [31], it follows that

$$\begin{aligned} |\operatorname{trace}(TL)| &\leqslant \pi_p(L'')\, \| K \|\, \| R \|\, i_{p'}(D) \\ &\leqslant \pi_p(L)[i_{p'}(T) + \epsilon]. \end{aligned}$$

The case $p = \infty$ is Lemma 2.1.

Proof of (b). Since $i_{p'} \leqslant \nu_{p'}$ we have $i_{p'}^\Delta \geqslant \nu_{p'}^\Delta$. Since $i_{p'}$ and $\nu_{p'}$ agree on operators between finite dimensional spaces $\nu_{p'}^\Delta = i_{p'}^\Delta = \pi_p$. Thus, $i_{p'}^\Delta \geqslant \nu_{p'}^\Delta \geqslant \nu_{p'}^* = \pi_p$.

If $T \in \mathscr{F}(E, F)$, let $L \in \mathscr{F}(F, E)$ and $\epsilon > 0$ be given. Factor L as in the following diagram

$$\begin{tikzcd} E \arrow[r, "T"] & F \arrow[r, "L"] \arrow[d, "A"'] & E \arrow[r, "I"] & E'' \\ & L_\infty(\mu) \arrow[rr, "B"'] & & L_{p'}(\mu) \arrow[u, "C"'] \end{tikzcd}$$

where $\| A \| = \| C \| = 1$, $i_{p'}(B) \leqslant (1 + \epsilon)\, i_{p'}(L)$ and I the natural injection. Then

$$\begin{aligned} |\operatorname{trace}(LT)| &= |\operatorname{trace}(ILT)| \\ &= |\operatorname{trace}(CBAT)|, \text{ and since } AT \text{ is finite rank} \\ &= |\operatorname{trace}((AT)''CB)|, \text{ and by Lemma 2.2} \\ &\leqslant i_1((AT)''CB) \\ &\leqslant \pi_p((AT)'')\, i_{p'}(CB) \\ &\leqslant \pi_p(T'')\, \| A \|\, \| C \|\, i_{p'}(B) \\ &\leqslant (1 + \epsilon)\, \pi_p(T)\, i_{p'}(L). \end{aligned}$$

This implies $i_{p'}^{\Delta}(T) \leqslant \pi_p(T)$. If E or F have m.a.p., by Proposition 2.4, $\pi_p^{\Delta\Delta}(T) = \pi_p(T)$ for all $T \in \mathscr{L}(E, F)$, but by (a) $\pi_p^{\Delta\Delta} = i_{p'}^{\Delta}$.

Proof of (c). We again have $\gamma_{p'} = j_p{}^* \leqslant j_p{}^{\Delta}$. For the other inequality, let $T \in \Gamma_{p'}(E, F)$, $\epsilon > 0$ and i_F the canonical embedding of F into F''. Let $L \in \mathscr{F}(F, E)$ and choose operators A and B so that $\| A \| \| B \| \leqslant (1 + \epsilon) \gamma_{p'}(T)$ and so that the following diagram commutes:

$$\begin{tikzcd} F \arrow[r, "L"] & E \arrow[r, "T"] \arrow[dr, "A"'] & F \arrow[r, "i_F"] & F'' \\ & & L_{p'}(\mu) \arrow[ur, "B"'] & \end{tikzcd}$$

As before trace(TL) = trace$((AL)'' B)$. Since L is finite rank and $L_{p'}(\mu)$ has m.a.p., $|\operatorname{trace}((AL)'' B)| \leqslant i_1((AL)'' B) = j_p((AL)'' B)$ by the remarks preceeding Table I.

$$\begin{aligned} |\operatorname{trace}(TL)| &\leqslant j_p((AL)''B) \\ &\leqslant j_p(L'') \| A \| \| B \| \\ &\leqslant (1 + \epsilon) j_p(L) \gamma_{p'}(T). \end{aligned}$$

The last inequality holds because j_p is perfect. Since L and ϵ were arbitrary, $j_p{}^{\Delta}(T) \leqslant \gamma_{p'}(T)$.

Proof of (d). Since $\gamma_{p'} \leqslant c_{p'}$ we have $\gamma_{p'}^{\Delta}(T) \geqslant c_{p'}^{\Delta}(T)$ and $c_{p'}^{\Delta}(T) \geqslant c_{p'}^{*}(T)$. But between finite dimensional spaces it is easily seen that $c_{p'} = \gamma_{p'}$; therefore, $c_{p'}^{*}(T) = \gamma_{p'}^{*}(T) = j_p(T)$ for all $T \in \mathscr{L}(E, F)$.

Suppose now that $T \in \mathscr{F}(E, F)$. Given $L \in \mathscr{F}(F, E)$ and $\epsilon > 0$, choose a factorization of L with $\| A \| \| B \| \leqslant (1 + \epsilon) \gamma_{p'}(L)$ so that the following diagram commutes:

$$\begin{tikzcd} E \arrow[r, "T"] & F \arrow[r, "L"] \arrow[dr, "A"'] & E \arrow[r, "I"] & E'' \\ & & L_{p'}(\mu) \arrow[ur, "B"'] & \end{tikzcd}$$

Then as in (b)

$$|\operatorname{trace}(LT)| = |\operatorname{trace}((AT)''B)| \leqslant i_1((AT)''B) = j_p((AT)''B)$$

(by Cohen's result [3]) $\leqslant j_p(T'') \| A \| \| B \| \leqslant (1 + \epsilon) j_p(T) \gamma_{p'}(L)$. Thus, $\gamma_{p'}^{\Delta}(T) \leqslant j_p(T)$.

If E or F has m.a.p. and $T \in \mathscr{L}(E, F)$, then by part (c) and Proposition 2.4

$$\gamma_{p'}^{\Delta}(T) = j_p^{\Delta\Delta}(T) = j_p(T).$$

COROLLARY 2.6. $\nu_p^{\Delta\Delta} = i_p^{\Delta\Delta} = i_p$ *and* $c_p^{\Delta\Delta} = \gamma_p^{\Delta\Delta} = \gamma_p$.

Proof. By (b) we get that $i_{p'}^{\Delta\Delta} = \nu_{p'}^{\Delta\Delta} = \pi_p{}^{\Delta}$, but $\pi_p{}^{\Delta} = i_{p'}$. Part (d) gives the second case.

Remarks. (1) Since $\pi_p(L) = \nu_p{}^Q(L)$ if $L \in \mathscr{F}(E, F)$ we obtain $\nu_p^{Q\Delta} = i_{p'} = \nu_p^{Q*}$. We used in the proof that $\gamma_{p'}^* = c_{p'}^* = j_p$, and that $\nu_{p'}^* = i_{p'}^* = \pi_p$, because $\nu_{p'} = i_{p'}$ between finite-dimensional spaces.

(2) The equality in (b) remains true for $T \in N_p{}^Q(E, F)$, which is seen by passing to the limit of finite rank operators in the $(\pi_p =)\ \nu_p{}^Q$-norm.

(3) Another version of (b), involving only a.p. in the case $1 < p < \infty$, will be given later.

The proof of the following lemma is straightforward.

LEMMA 2.7. *If* $1 \leqslant p,\ r < \infty$, $q^{-1} = r^{-1} + p^{-1} \leqslant 1$ *and* a, b *are positive reals, then the function* $\varphi(t) = p^{-1}t^p a^p + r^{-1}t^{-r}b^r$ *has a minimum value of* $q^{-1}(ab)^q$ *at some point in* $(0, \infty)$.

THEOREM 2.8. *For* $1 \leqslant q \leqslant p \leqslant \infty$, I_{pq} *and* Γ_{pq} *are Banach ideals.*

Proof. The only thing that need be shown is that γ_{pq} and i_{pq} both satisfy the triangle inequality. For the sake of simplicity we give only the proof for γ_{pq} when $1 < q < p < \infty$.

Let $T_i \in \Gamma_{pq}(E, F)$ for $i = 1, 2$, and let I be the natural inclusion of F into F''. Let $A_i: E \to L_p(\mu_i)$, $B_i: L_p(\mu_i) \to L_q(\mu_i)$ and $C_i: L_q(\mu_i) \to F''$ be any operators satisfying $C_iB_iA_i = IT_i$, $i = 1, 2$. Let ν be the measure on the disjoint union of S_1 and S_2 which agrees with μ_i on S_i . Then clearly $L_p(\nu) = (L_p(\mu_1) \oplus L_p(\mu_2))_p$ and $L_q(\nu) = (L_q(\mu_1) \oplus L_q(\mu_2))_q$. For positive numbers s, t, u, and v define $A: E \to L^p(\nu)$ by $A(x) = (uvA_1(x), stA_2(x))$, $B: L^p(\nu) \to L^q(\nu)$ by $B(f, g) = (u^{-1}B_1(f), s^{-1}B_2(g)$, and $C: L^q(\nu) \to F''$ by $C(f, g) = v^{-1}C_1(f) + t^{-1}C_2(g)$. Then clearly $CBA = I(T_1 + T_2)$ and

$$\| A \| \leqslant [(uv)^p \| A_1 \|^p + (st)^p \| A_2 \|^p]^{1/p},$$

$$\| B \| \leqslant [u^{-r} \| B_1 \|^r + s^{-r} \| B_2 \|^r]^{1/r}, \qquad q^{-1} = p^{-1} + r^{-1},$$

and

$$\| C \| \leqslant [v^{-q'} \| C_1 \|^{q'} + t^{-q'} \| C_2 \|^{q'}]^{1/q'}.$$

By Lemma 2.7,

$$\begin{aligned}\gamma_{pq}(T_1 + T_2) &\leqslant \| A \| \| B \| \| C \| \\ &\leqslant q^{-1}(\| A \| \| B \|)^q + (q')^{-1} \| C \|^{q'} \\ &\leqslant p^{-1} \| A \|^p + r^{-1} \| B \|^r + (q')^{-1} \| C \|^{q'} \\ &\leqslant p^{-1} u^p (v \| A_1 \|)^p + r^{-1} u^{-r} \| B_1 \|^r \\ &\quad + p^{-1} s^p (t \| A_2 \|)^p + r^{-1} s^{-r} \| B_2 \|^r \\ &\quad + (q')^{-1} \| C \|^{q'}.\end{aligned}$$

Taking the infimum successively over all positive u and s gives

$$\begin{aligned}\gamma_{pq}(T_1 + T_2) \leqslant q^{-1}(v \| A_1 \| \| B_1 \|)^q + (q')^{-1} v^{-q'} \| C_1 \|^{q'} \\ + q^{-1}(t \| A_2 \| \| B_2 \|) + (q')^{-1} t^{-q'} \| C_2 \|^{q'}.\end{aligned}$$

Another two applications of the lemma shows

$$\gamma_{pq}(T_1 + T_2) \leqslant \| A_1 \| \| B_1 \| \| C_1 \| + \| A_2 \| \| B_2 \| \| C_2 \|.$$

Finally, taking the infimum over all possible factorizations of T_1 and T_2 yields the triangle inequality.

When $p = q$ it is clear that $I_{pq} = \Gamma_{pq} = \Gamma_p$. A simple description of I_{pq} for $q < p$ is given in the following result.

Proposition 2.9. *Let $1 \leqslant q < p \leqslant \infty$, $T \in I_{pq}(E, F)$ and $\epsilon > 0$. There is a probability measure ν and operators $A: E \to L_p(\nu)$, $B: L_q(\nu) \to F''$ such that $\| A \| \| B \| \leqslant (1 + \epsilon)\, i_{pq}(T)$ and $IT = BJA$, where $I: F \to F''$ and $J: L_p(\nu) \to L_q(\nu)$ are the natural injections.*

Proof. For convenience assume $p < \infty$. Let $Q: E \to L_p(\mu)$, $R: L_p(\mu) \to L_q(\mu)$ and $S: L_q(\mu) \to F''$ satisfy $IT = SRQ$ and $\| S \| \| R \| \| Q \| \leqslant (1 + \epsilon)\, i_{pq}(T)$, where R is diagonal multiplication by $f \in L_r(\mu)$, $q^{-1} = r^{-1} + p^{-1}$. Let $d\nu = \| f \|^{-r} | f |^r \, d\mu$, define $A: E \to L_p(\nu)$ by $A(x) = \operatorname{sgn}(f) \, | f |^{-r/p} Q(x)$ and $B: L_q(\nu) \to F''$ by $B(g) = S(| f |^{r/q} g)$. It is easy to check that $BJA = IT$ and that $\| A \| \leqslant \| Q \| \| f \|^{-r/p}$, $\| B \| \leqslant \| S \| \| f \|^{r/q}$. Since $\| R \| = \| f \|$, this gives the inequality for the i_{pq} norm of T.

Notice that from the proposition, $I_{\infty p}$ is exactly the ideal I_p of all p-integral operators, and that I_{p1} is the dual ideal $I_{p'}$ of the p'-integral operators.

THEOREM 2.10. *For $1 \leqslant q \leqslant p \leqslant \infty$, Γ_{pq} is a perfect ideal.*

Proof. The proof given here involves some complications of the arguments of [22]. Suppose $1 \leqslant q \leqslant p < \infty$, and let $T \in \mathscr{L}(E, F)$. By Proposition 1.3 it is enough to prove that if $\gamma_{pq}(UTV) \leqslant b$ for all finite rank U and V with norms at most one, then $\gamma_{pq}(T) \leqslant b$.

Throughout the proof $B(E')$ and $B(F'')$ are the spaces of bounded, real valued functions on the closed unit balls of E' and F'', respectively. For $x \in E$ and $y' \in F'$, f_x and $g_{y'}$ are the elements of $B(E')$ and $B(F'')$ naturally associated with x and y', respectively. The unit vector bases in l_p and l_q are denoted by $(e_i, e_i')_{i \geqslant 1}$ and $(b_k, b_k')_{k \geqslant 1}$. Z is the 1-point compactification of the real numbers. $\mathscr{D}$ is the set of all ordered pairs (N, M), where $N \subset E$ and $M \subset F'$ are finite dimensional subspaces. $\mathscr{D}$ is directed by declaring that $(N, M) \leqslant (N_1, M_1)$ iff $N \subset N_1$ and $M \subset M_1$. We now proceed to define a net

$$\{\Phi_{(N,M)}\}_{(N,M) \in \mathscr{D}} \subset (Z \times Z \times Z)^{B(E') \times B(F'')}$$

in the following manner.

Given $(N, M) \in \mathscr{D}$, let V_N be the inclusion of N into E and U_M be the restriction map from F to M'. Then $\gamma_{pq}(U_M T V_N) \leqslant b$. Since each $L_t(\mu)$ space is $\mathscr{L}_{t,1+\epsilon}$ for all $\epsilon > 0$, there are natural numbers n and m, and operators $A_{(N,M)}\colon N \to l_p^n$, $B_{(N,M)}\colon l_p^n \to l_q^m$, and $C_{(N,M)}\colon l_q^m \to M'$ such that

$$C_{(N,M)} B_{(N,M)} A_{(N,M)} = U_M T V_N, \qquad \| A_{(N,M)} \| \leqslant 1, \qquad \| C_{(N,M)} \| \leqslant 1$$

and

$$\| B_{(N,M)} \| \leqslant b[1 + \dim N^{-1} + \dim M^{-1}].$$

We will write $b(N, M)$ for the last displayed constant. For $i = 1, 2, \ldots, n$, let $x_i' \in E'$ be norm preserving extensions of the functionals $A_{(N,M)}(e_i')$. Similarly, let $y_k'' \in F''$ be norm preserving extensions of $C_{(N,M)}(b_k)$, $k = 1, 2, \ldots, m$. Let $Q_{(N,M)}$ be the semi-norm on $B(E')$ defined by

$$Q_{(N,M)}(f) = \Big(\sum_{i \leqslant n} |f(x_i')|^p\Big)^{1/p},$$

$S_{(N,M)}$ the seminorm on $B(F'')$ defined by

$$S_{(N,M)}(g) = \Big(\sum_{k \leqslant m} |g(y_k'')|^{q'}\Big)^{1/q'}$$

and $R_{(N,M)}$ the bilinear form on $B(E') \times B(F'')$ given by

$$R_{(N,M)}(f, g) = \left\langle B_{(N,M)}\left(\sum_{i\leqslant n} f(x_i')\, e_i\right), \sum_{k\leqslant m} g(y_k'')\, b_k'\right\rangle.$$

Now observe that

(a) $Q_{(N,M)}(f_x) \leqslant \| x \|$ for $x \in N$;

(b) $S_{(N,M)}(g_{y'}) \leqslant \| y' \|$ for $y' \in M$;

(c) $| R_{(N,M)}(f, g)| \leqslant b(N, M)\, Q_{(N,M)}(f)\, S_{(N,M)}(g)$ for $f \in B(E')$, $g \in B(F''')$; and

(d) $R_{(N,M)}(f_x, g_{y'}) = \langle Tx, y'\rangle$ for $x \in N$, $y' \in M$.

Finally, let $\Phi_{(N,M)}$ be the function on $B(E') \times B(F'')$ defined by $\Phi_{(N,M)}(f, g) = (Q_{(N,M)}(f), R_{(N,M)}(f, g), S_{(N,M)}(g))$.

The net so defined lies in a compact topological space (topology of simple convergence) so the filter sections of the net refine the neighborhood basis at some point Φ in the space. Let Q, R and S be the functions given by $\Phi(f, g) = (Q(f), R(f, g), S(g))$. It is easily seen that Q and S are extended real valued semi-norms on $B(E')$ and $B(F'')$, respectively, and that from (a) and (b) the inequalities $Q(f_x) \leqslant \| x \|$, $S(g_{y'}) \leqslant \| y' \|$ are valid for all $x \in E$, $y' \in F'$. Set

$$X_p = \{f \in B(E') : Q(f) < \infty\},$$

and define $Y_{q'}$ similarly using S. Then X_p is a vector subspace and solid sublattice of $B(E')$ under the pointwise operations and relations. Further $Q(f \pm g)^p = Q(f)^p + Q(g)^p$ whenever $\min(f, g) = 0$ since each semi-norm $Q_{(N,M)}$ has this property. Write $eq(f)$ for the equivalence class in $X_p/X_p \cap Q^{-1}(0)$ which contains f. By Nakano's theorem [26], the completion of this space under the norm $\| eq(f)\| = Q(f)$ is isometric to a space $L_p(\mu)$. Similarly, the completion of $Y_{q'}/Y_{q'} \cap S^{-1}(0)$ is isometric to some $L_{q'}(\nu)$-space.

From (c) we have that $| R(f, g)| \leqslant bQ(f)\, S(g)$ for all $f \in X_p$, $g \in Y_{q'}$, and from (d) that $R(f_x, g_{y'}) = \langle Tx, y'\rangle$ whenever $x \in E$, $y' \in F'$. Define $A: E \to L_p(\mu)$ by $A(x) = eq(f_x)$, $C: F' \to L_{q'}(\nu)$ by $C(y') = eq(g_{y'})$ and $B: L_p(\mu) \to L_{q'}(\nu)' = L_q(\nu)$ so that

$$\langle eq(g), B(eq(f))\rangle = R(f, g).$$

Then $\| A \| \leqslant 1$, $\| C \| \leqslant 1$, and $\| B \| \leqslant b$. Further, if I is the natural inclusion of F into F'', then $IT = C'BA$. This completes the proof.

We now discuss the ideal J_{pq}. It will follow from Theorem 2.15 that j_{pq} is a norm. However, it is clear that $j_{pq}(UTV) \leqslant \| U \|\, j_{pq}(T)\, \| V \|$

for all U, T and V. This fact will be used without further mention in the following two theorems.

THEOREM 2.11. *For* $1 \leqslant q \leqslant p \leqslant \infty$, $S \in I_{pq}(E,F)$ *and*

$$T \in J_{q'p'}(F, G),\ i_1(TS) \leqslant i_{pq}(S) j_{q'p'}(T).$$

If $1 < p < \infty$, *then* $\nu_1(TS) \leqslant i_{pq}(S)\, j_{q'p'}(T)$.

Proof. For $q = p$ this is given in [3]. Assume $q < p$. From the ideal structure of the operators involved we need only prove the theorem in the case $E = L_p(\mu)$, $F = L_q(\mu)$ and S a diagonal map. By Proposition 2.9 we may assume that μ is a probability measure and that S is injection. Let $\pi = (D_i)_{i \leqslant n}$ be an arbitrary decomposition of the space underlying μ; that is, π is a pairwise disjoint collection of sets of positive finite μ-measure. The natural norm one projection onto $[\chi_{D_i}]_{i \leqslant n}$ is given by $P_\pi(f) = \sum_{i \leqslant n} \mu(D_i)^{-1} \langle f, \chi_{D_i} \rangle \chi_{D_i}$. It will first be shown that $i_1(TSP_\pi) \leqslant j_{q'p'}(T)$. Define maps $A: L_p(\mu) \to l_p^n$, $B: l_p^n \to l_q^n$, and $C: l_q^n \to L_q(\mu)$ by $A(f) = (\mu(D_i)^{-1/p'} \langle f, \chi_{D_i} \rangle)_{i \leqslant n}$, $B((x_i)_{i \leqslant n}) = (x_i \mu(D_i)^{1/r})_{i \leqslant n}$ and $C((x_i)_{i \leqslant n}) = \sum_{i \leqslant n} x_i \mu(D_i)^{-1/q} \chi_{D_i}$, where $q^{-1} = r^{-1} + p^{-1}$. It is easily checked that A, B, and C are contractions and that $SP_\pi = CBA$. To calculate the integral norm of TCB, let $U: G \to l_p^n$ be an arbitrary operator, represented by $(g_k')_{k \leqslant n} \subset G'$. Then

$$\operatorname{trace}(UTCB) = \sum_{i \leqslant n} \mu(D_i)^{-1/p} \langle T(\chi_{D_i}), g_i' \rangle.$$

Let $IT = RQ$, where I is the natural embedding of G into G'', Q is q'-summing, and R' is p-summing. Writing $f_i = \mu(D_i)^{-1/q} \chi_{D_i}$ we have that

$$\begin{aligned}
|\operatorname{trace}(UTCB)| &\leqslant \sum_{i \leqslant n} \mu(D_i)^{1/r} |\langle Q(f_i), R'(g_i') \rangle| \\
&\leqslant \Big(\sum_{i \leqslant n} \mu(D_i)\Big)^{1/r} \Big(\sum_{i \leqslant n} |\langle Q(f_i), R'(g_i') \rangle|^{r'}\Big)^{1/r'} \\
&\leqslant \Big(\sum_{i \leqslant n} \| Q(f_i) \|^{q'}\Big)^{1/q'} \Big(\sum_{i \leqslant n} \| R'(g_i') \|^{p}\Big)^{1/p} \\
&\leqslant \pi_{q'}(Q)\, \epsilon_{q'}(f_i)\, \pi_p(R')\, \epsilon_p(g_i') \\
&\leqslant \pi_{q'}(Q)\, \pi_p(R') \| U \|.
\end{aligned}$$

Taking the supremum over $\| U \| \leqslant 1$ and the infimum over factorizations of T gives $i_1(TCB) \leqslant j_{q'p'}(T)$, so that also $i_1(TSP_\pi) = i_1(TCBA) \leqslant \| A \|\, j_{q'p'}(T) \leqslant j_{q'p'}(T)$.

It is well known [5] that the net (P_π), directed in the usual manner, converges simply to the identity on $L_p(\mu)$. Thus, (TSP_π) converges simply to TS, and so TS is integral with $i_1(TS) \leqslant j_{q'p'}(T)$. For $1 < p < \infty$, the results of Theorem 10 of [12], shows that $\nu_1(TS) = i_1(TS)$.

Using the factorization definitions and a standard duality argument it is easy to check that $i_{pq}(T') = i_{q'p'}(T)$ and $j_{pq}(T') = j_{q'p'}(T)$ for every operator T. From [12, Proposition 1.5], it is known that an operator is integral iff its adjoint is integral, and that an operator with nuclear adjoint is nuclear into the bidual of its range.

COROLLARY 2.12. *For* $1 \leqslant q \leqslant p \leqslant \infty$, $T \in J_{q'p'}(E, F)$ *and* $S \in I_{pq}(F, G)$, $i_1(ST) \leqslant i_{pq}(S)\, j_{q'p'}(T)$. *If* $1 < q < \infty$, *then* ST *is nuclear as an operator from* E *into* G'', *and* $\nu_1(ST) \leqslant i_{pq}(S)\, j_{q'p'}(T)$.

COROLLARY 2.13. *If* $1 < p < \infty$, $S \in I'_{p'}(E, F)$ *and* $T \in \Pi_p'(F, G)$, *then* $\nu_1(TS) \leqslant i_{p'}(S')\, \pi_p(T')$.

Proof. By Theorem 2.11 $\nu_1(TS) \leqslant i_{p1}(S)\, j_{\infty p'}(T)$. But $i_{p1} = i_{\infty p'} = i_{p'}$, and $j_{\infty p'} = \pi_p'$ by taking stars (*).

COROLLARY 2.14. *Let* $1 < q < \infty$, $S \in I_q'(F, G)$, $T \in \Pi'_{p'}(E, F)$ *and* $i\colon G \to G''$ *be the inclusion map. Then* $iST \in N_1(E, G'')$ *and*

$$\nu_1(iST) \leqslant i_q(S')\, \pi_{q'}(T').$$

THEOREM 2.15. *For* $1 \leqslant q \leqslant p \leqslant \infty$, $i_{pq}^* = j_{q'p'}$.

Proof. The first claim is that $i_{pq}^*(T) \leqslant j_{q'p'}(T)$ for $T \in \mathscr{L}(E, F)$. Let X and Y be finite dimensional spaces, $U \in \mathscr{L}(X, E)$, $V \in \mathscr{L}(F, Y)$ and $S \in \mathscr{L}(Y, X)$. Then by Lemma 2.2 and Theorem 2.11

$$|\operatorname{trace}(SVTU)| \leqslant i_1(SVTU) \leqslant i_{pq}(SV) j_{q'p'}(TU) \leqslant i_{pq}(S) \| U \| \| V \| j_{q'p'}(T).$$

By definition this gives $i_{pq}^*(T) \leqslant j_{q'p'}(T)$.

Now suppose $T \in I_{pq}^*(E, F)$. For $(x_i)_{i \leqslant n} \subset E$, $(y_i')_{i \leqslant n} \subset F'$ and $a = (a_i)_{i \leqslant n} \in l_r^n$, $q^{-1} = r^{-1} + p^{-1}$, let $U\colon l_q^n \to E$ and $V\colon F \to l_p^n$ be the operators defined by the series, and $R\colon l_p^n \to l_q^n$ be diagonal multiplication by $(a_i)_{i \leqslant n}$. Then

$$\begin{aligned} \Big| \sum_{i \leqslant n} a_i \langle Tx_i, y_i' \rangle \Big| &= |\operatorname{trace}(RVTU)| \\ &\leqslant i_{pq}^*(T) \| U \| \| V \| i_{pq}(R) \\ &\leqslant i_{pq}^*(T)\, \epsilon_{q'}(x_i)\, \epsilon_p(y_i') \| a \|_r . \end{aligned}$$

Taking the supremum over $\| a \|_r \leqslant 1$ gives

$$\Big(\sum_{i\leqslant n} |\langle Tx_i, y_i'\rangle|^{r'}\Big)^{1/r'} \leqslant i_{pq}^*(T)\, \epsilon_{q'}(x_i)\, \epsilon_p(y_i').$$

By Lemma 2.7

$$\sum_{i\leqslant n} |\langle Tx_i, y_i'\rangle|^{r'} \leqslant i_{pq}^*(T)^{r'}\, [(r'/q')\, \epsilon_{q'}(x_i)^{q'} + (r'/p)\, \epsilon_p(y_i')^p]. \tag{1}$$

Let $K_1 \subset E'$ and $K_2 \subset F''$ be the weak-star closures of the extreme points of the closed unitballs and set $K = K_1 \times K_2$. For $(x, y') \in E \times F'$ let $\varphi_{(x,y')}$ be the element of $C(K)$ given by

$$\varphi_{(x,y')}(x', y'') = (r'/q')\, |\langle x, x'\rangle|^{q'} + (r'/p)\, |\langle y', y''\rangle|^p.$$

Let A be the closed convex hull of $\{\varphi_{(x,y')} : |\langle Tx, y'\rangle| = 1\}$ and $B = \{f \in C(K) : f(t) < i_{pq}^*(T)^{-r'}$ for all $t \in K\}$. Then A and B are disjoint convex sets with B open; in fact, if $\varphi_{(x_i,y_i')}$ satisfy $|\langle Tx_i, y_i'\rangle| = 1$, $i = 1, 2,..., n$, $0 \leqslant \lambda_i \leqslant 1$ and $\sum_{i\leqslant n} \lambda_i = 1$, then by (1)

$$\begin{aligned} i_{pq}^*(T)^{-r'} &= i_{pq}^*(T)^{-r'} \sum_{i\leqslant n} \lambda_i\, |\langle Tx_i, y_i'\rangle|^{r'} \\ &\leqslant \Big\| \sum_{i\leqslant n} \lambda_i \varphi_{(x_i,y_i')} \Big\|_{C(K)}. \end{aligned}$$

By the separation theorem for convex sets and the Riesz theorem, there is a probability measure σ on K such that $i_{pq}^*(T)^{-r'} \leqslant \sigma(f)$ for $f \in A$. This gives

$$i_{pq}^*(T)^{-r'} \leqslant (r'/q') \int_K |\langle x, t\rangle|^{q'}\, \sigma(dt, ds) + (r'/p) \int_K |\langle y', s\rangle|^p\, \sigma(dt, ds),$$

whenever $|\langle Tx, y'\rangle| = 1$. Replacing x, y' by tx, $t^{-1}y'$ and applying Lemma 2.7 gives

$$i_{pq}^*(T)^{-r'} \leqslant \Big(\int_K |\langle x, t\rangle|^{q'}\, d\sigma\Big)^{r'/q'} \Big(\int_K |\langle y', s\rangle|^p\, d\sigma\Big)^{r'/p},$$

whenever $|\langle Tx, y'\rangle| = 1$, which in turn implies

$$|\langle Tx, y'\rangle| \leqslant i_{pq}^*(T) \Big(\int_K |\langle x, t\rangle|^{q'}\, d\sigma\Big)^{1/q'} \Big(\int_K |\langle y', s\rangle|^p\, d\sigma\Big)^{1/p}$$

for all $x \in E$, $y' \in F'$. Arguing exactly as in [18] shows that $j_{q'p'}(T) \leqslant i_{pq}^*(T)$, which completes the proof.

For later use we give two corollaries to the proof of the theorem.

COROLLARY 2.16. *Let $1 \leqslant q \leqslant p \leqslant \infty$, $b > 0$ and $T \in \mathscr{L}(E, F)$. Then $j_{pq}(T) \leqslant b$ iff, for all finite sets $(x_i)_{i \leqslant n} \subset E$ and $(y_i')_{i \leqslant n} \subset F'$,*

$$\left(\sum_{i \leqslant n} |\langle Tx_i, y_i' \rangle|^r\right)^{1/r} \leqslant b\epsilon_p(x_i)\, \epsilon_{q'}(y_i'),$$

where $r^{-1} = (q')^{-1} + p^{-1}$. Further, $j_{pq}(T)$ is the smallest such constant b.

COROLLARY 2.17. *Let $1 \leqslant q \leqslant p \leqslant \infty$, $b > 0$, $T \in \mathscr{L}(E, F)$, $K_1 \subset E'$ and $K_2 \subset F''$ the weak-star closures of the extreme points of the closed unit balls. Then $j_{pq}(T) \leqslant b$ iff there are normalized Radon measure μ and ν on K_1 and K_2 such that, for all $x \in E$ and $y' \in F'$,*

$$|\langle Tx, y' \rangle| \leqslant b \left(\int_{K_1} |\langle x, t \rangle|^p \mu(dt)\right)^{1/p} \left(\int_{K_2} |\langle y', s \rangle|^{q'} \nu(ds)\right)^{1/q'}.$$

Further, $j_{pq}(T)$ is the smallest suc constant b.

COROLLARY 2.18. *J_{pq} is a perfect ideal.*

Proof. Every adjoint ideal is perfect.

COROLLARY 2.19. *Let $T \in \mathscr{L}(E, F)$ and $1 \leqslant q \leqslant p \leqslant \infty$. If $1 < p < \infty$ and F has a.p., then $i_{pq}^{\Delta}(T) = i_{pq}^*(T)$. If $1 < q < \infty$ and E' has a.p., then $i_{pq}^{\Delta}(T) = i_{pq}^*(T)$.*

Proof. In the first case let $S \in \mathscr{F}(F, E)$. By Theorem 2.11, $\nu_1(TS) \leqslant i_{pq}(S)\, j_{q'p'}(T) = i_{pq}(S)\, i_{pq}^*(T)$. Since F has a.p. $|\operatorname{trace}(TS)| \leqslant \nu_1(TS)$, which gives $i_{pq}^{\Delta}(T) \leqslant i_{pq}^*(T)$. The other case is similar.

Remark. (1) The preceeding corollary and the relations $I_{\infty p} = I_p$, $I_{pp} = \Gamma_p$ give sharpened versions of Theorem 2.5. (2). For $T \in \mathscr{F}(E, F)$ and $1 \leqslant q \leqslant p \leqslant \infty$, one can prove $i_{pq}^{\Delta}(T) = i_{pq}^*(T)$ using the method of Theorem 2.5(d). The difference is that the proof uses Theorem 2.11 instead of the cited result of Cohen. It follows then that $i_{pq}^{\Delta\Delta} = j_{q'p'}^{\Delta}$.

PROPOSITION 2.20. *Let $T \in \mathscr{L}(E, F)$, c be a constant and suppose $i_1(ST) \leqslant c\alpha(S)$ for every finite dimensional G and $S \in A(F, G)$. Then $\alpha^*(T) \leqslant c$.*

Proof. Let X, Y be finite dimensional, $U \in \mathscr{L}(X, E)$, $V \in \mathscr{L}(F, Y)$ and $S \in A(Y, X)$. Then

$$\begin{aligned} |\operatorname{trace}(USVT)| &\leqslant \| U \| \, \| SVT \|^{\Delta} \\ &= \| U \| \, i_1(SVT) \\ &\leqslant c \, \| U \| \, \alpha(SV) \\ &\leqslant c \, \| U \| \, \| V \| \, \alpha(S), \end{aligned}$$

so that $\alpha^*(T) \leqslant c$ by the definition.

COROLLARY 2.21. *Let $T \in \mathscr{L}(E, F)$ and suppose that $A(F, G)$ is a Banach space for every G. If $ST \in I_1(E, G)$ whenever $S \in A(F, G)$, then $T \in A^*(E, F)$.*

The proof follows from the well known technique of Nachbin [25].

Another variant of Proposition 2.20 is Proposition 2.22.

PROPOSITION 2.22. *If $T \in \mathscr{L}(E, F)$, E has m.a.p. and $i_1(ST) \leqslant C\alpha(S)$ for all $S \in \mathscr{F}(F, E)$ then $\alpha^{\Delta}(T) \leqslant C$.*

Proof. By Lemma 2.2, $|\operatorname{trace}(ST)| \leqslant i_1(ST) \leqslant C\alpha(S)$; hence, $\alpha^{\Delta}(T) \leqslant C$.

Although I_{pq} may be nonperfect,[1] we have the following theorem.

THEOREM 2.23. *Consider the following statements, where $1 \leqslant q \leqslant p \leqslant \infty$:*

(a) $T \in I_{pq}(E, F)$;

(b) *For every G and $S \in \Pi_{q'}(F, G)$, $ST \in I_{p1}(E, G)$;*

(c) $T \in I_{pq}^{**}(E, F)$.

Then (a) *implies* (b) *and* (b) *implies* (c), *and always $i_{p1}(ST) \leqslant \pi_{q'}(S) \, i_{pq}(T)$.*

Proof. Suppose $T \in I_{pq}(E, F)$ and let $S \in \Pi_{q'}(F, G)$. For any $R \in I_{p1}^*(G, H) = J_{\infty p'}(G, H)$, $RS \in J_{q'p'}(F, H)$. By Theorem 2.11,

$$R(ST) \in I_1(E, H) \quad \text{and} \quad i_1(R(ST)) \leqslant j_{q'p'}(RS) \, i_{pq}(T) \leqslant i_{p1}^*(R) \, \pi_{q'}(S) \, i_{pq}(T)$$

By Proposition 2.20, $i_{p1}(ST) \leqslant \pi_{q'}(S) \, i_{pq}(T)$.

Let $U \in I_{pq}^*(F, H) = J_{q'p'}(F, H)$, and factor $IU = VS$, $S \in \Pi_{q'}(F, G)$, $V \in D_{p'}(G, H'')$ and I the embedding of H into H''. By (b) $ST \in I_{p1}(E, G)$, hence, $(ST)' \in I_{p'}(G', E')$ and $(ST)' \, V' \in I_1(H''', E')$, which implies $VST \in I_1(E, H'')$. Again from Proposition 2.20, $T \in I_{pq}^{**}(E, F)$.

THEOREM 2.24. *If E' or F has m.a.p. and $T \in C_p(E, F)$, then $c_p(T) = \gamma_p(T)$.*

Proof. The inequality $\gamma_p(T) \leqslant c_p(T)$ always holds. First consider the case in which T is a finite rank operator. Let I be the natural embedding of F into F'' and write $IT = VU$, where $U \in \mathscr{L}(E, L_p(\mu))$ and $V \in \mathscr{L}(L_p(\mu), F'')$.

Assume that F has m.a.p., and let $\epsilon > 0$. By [16] there is an $R \in \mathscr{F}(F, F)$ satisfying $\| R \| < 1 + \epsilon$ and $RT = T$. Then $R''V$ is a compact operator from $L_p(\mu)$ to F, so by [15] $R''V = BA$, where $A \in C(L_p(\mu), l_p)$, $B \in C(l_p, F)$ and $\| A \| \| B \| \leqslant (1 + \epsilon) \| R''V \|$. Then $T = B(AU)$ and $c_p(T) \leqslant \| B \| \| AU \| \leqslant (1 + \epsilon)^2 \| V \| \| U \|$. Since the factorization and $\epsilon > 0$ were arbitrary, $c_p(T) \leqslant \gamma_p(T)$.

Now assume that E' has m.a.p. and let $\epsilon > 0$. As before choose $R \in \mathscr{F}(E', E')$ so that $RT' = T'$ and $\| R \| < 1 + \epsilon$. Write J for the natural embedding of E into E'', P for the restriction from F'''' onto F'', $A = PV''$ and $B = (RU')' J$. Then $IT = AB$, $\| A \| \| B \| \leqslant (1 + \epsilon) \| V \| \| U \|$ and B is finite rank. $B(E)$ is a finite dimensional subspace of the $\mathscr{L}_{p,1+\epsilon}$-space $L_p(\mu)''$, so there is a finite dimensional G, $B(E) \subset G \subset L_p(\mu)''$, and an isomorphism $S: G \to l_p{}^n$ with $\| S \| \| S^{-1} \| < 1 + \epsilon$. By the local reflexivity principle [23] there is an isomorphism $Q: A(G) \to F$ with $\| Q \| < 1 + \epsilon$ and $QIy = y$ for all $y \in I(F) \cap A(G)$. Finally we have that

$$T = (QAS^{-1})(SB) \quad \text{and} \quad c_p(T) \leqslant \| QAS^{-1} \| \| SB \| \leqslant (1 + \epsilon)^3 \| V \| \| U \|,$$

as before this means that $c_p(T) \leqslant \gamma_p(T)$.

An arbitrary $T \in C_p(E, F)$ factors as $T = AB$, $B \in C(E, l_p)$ and $A \in C(l_p, F)$. Let P_n be the projection of l_p onto the span of the first n unit vectors. Certainly the sequence (AP_nB) converges to T in both $c_p(\cdot)$ and $\gamma_p(\cdot)$ norm, and by the preceeding paragraphs $c_p(AP_nB) = \gamma_p(AP_nB)$ for each n. Passing to the limit gives $c_p(T) = \gamma_p(T)$.

COROLLARY 2.25. *If E' has m.a.p., $T \in C_p(E, F)$ and $S \in J_{p'}(F, G)$, then ST is nuclear and $\nu_1(ST) \leqslant c_p(T) j_{p'}(S)$.*

Proof. By Theorem 2.11 we have that ST is integral and that $i_1(ST) \leqslant \gamma_p(T) j_{p'}(S)$, and, hence, $i_1(ST) \leqslant c_p(T) j_{p'}(S)$ by Theorem 2.24. Fix S and consider the mapping from $C_p(E, F)$ into $I_1(E, G)$ which sends T to ST. By the above remarks this mapping is continuous. Since E' has m.a.p., $N_1(E, G) \subset I_1(E, G)$ isometrically and it is clear that $T \to ST$ sends each finite rank S to an element of $N_1(E, G)$.

But the finite rank operators are dense in $C_p(E, F)$, so $ST \in N_1(E, G)$ for every S, and $\nu_1(ST) = i_1(ST)$.

Similar reasoning establishes the following corollary.

COROLLARY 2.26. *If F has m.a.p., $S \in J_{p'}(H, E)$ and $T \in C_p(E, F)$, then TS is nuclear and $\nu_1(TS) \leqslant c_p(T)\, j_{p'}(S)$.*

From Theorem 2.24 above we gain a description of the dual of $C_p(E, F)$. Each functional $\varphi \in C_p(E, F)'$ naturally defines an operator $S \in \mathscr{L}(F, E'')$ by $\langle x', Sy\rangle = \langle x' \otimes y, \varphi\rangle$.

THEOREM 2.27. *If E' or F has m.a.p., then $C_p(E, F)' = J_{p'}(F, E'')$ naturally and isometrically. The action is given by*

$$\langle T, S\rangle = \operatorname{trace}(ST), \qquad \textit{if} \quad E' \textit{ has m.a.p.};$$

$$\langle T, S\rangle = \operatorname{trace}(T''S) \qquad \textit{if} \quad F \textit{ has m.a.p.}$$

(In the first case $ST \in N_1(E, E'')$, and in the second $T''S \in N_1(F, F)$.)

Proof. We give the proof when F has m.a.p. Let $S \in J_{p'}(F, E'')$. Then for every $T \in C_p(E, F)$ we have by Theorem 2.5(c) and Lemma 2.3 that $T''S \in I_1(F, F'')$ and $i_1(T''S) \leqslant \gamma_p(T'')\, j_{p'}(S)$. But $\gamma_p(T'') = \gamma_p(T)$ and $\gamma_p(T) = c_p(T)$ by Theorem 2.24. The operator T is compact so $T''S$ maps into F, and so $T''S \in I_1(F, F)$ and $i_1(T''S) \leqslant c_p(T)\, j_{p'}(S)$. Consider the operator from $C_p(E, F)$ to $I_1(F, F)$ which maps T to $T''S$. Since F has m.a.p. $N_1(F, F) \subset I_1(F, F)$ isometrically. Certainly the map $T \to T''S$ sends each finite rank T to an element of $N_1(F, F)$. But then by continuity and the density of the finite rank operators in $C_p(E, F)$, we have that $T''S \in N_1(F, F)$ for every $T \in C_p(E, F)$, and further that $\nu_1(T''S) \leqslant c_p(T)\, j_{p'}(S)$. This means that the second displayed formula is meaningful (F has a.p.), and that the formula defines a functional $\varphi \in C_p(E, F)'$ which satisfies $\| \varphi \| \leqslant j_{p'}(S)$.

To complete the proof it is necessary to show that given $\varphi \in C_p(E, F)'$, there is an $S \in J_{p'}(F, E'')$ which represents φ by the second displayed formula and which satisfies $j_{p'}(S) \leqslant \| \varphi \|$. Given φ, define S by $\langle x' \otimes y, \varphi\rangle = \langle x', Sy\rangle$. For $(y_i)_{i\leqslant n} \subset F$ and $(x_i')_{i\leqslant n} \subset E'$,

$$\begin{aligned} \Big| \sum_{i\leqslant n} \langle x_i', Sy_i\rangle \Big| &= |\langle R, \varphi\rangle| \\ &\leqslant \| \varphi \|\, \epsilon_p(x_i')\, \epsilon_{p'}(y_i), \end{aligned}$$

where $R = \sum_{i\leqslant n} x_i' \otimes y_i$. By [3] this means that $1 \otimes S$ induces a mapping from $l_{p'} \check{\otimes} F$ into $l_{p'} \hat{\otimes} E'' \subset (l_p \check{\otimes} E')'$ of norm at most $\| \varphi \|$, so that $j_{p'}(S) \leqslant \| \varphi \|$. Finally, φ and the functional defined by S agree on finite rank elements of $C_p(E, F)$, so by the continuity of both functionals, $\langle T, \varphi \rangle = \text{trace}(T''S)$ for all $T \in C_p(E, F)$.

COROLLARY 2.28. *Let $T \in \mathscr{L}(E, F)$ and I be the natural embedding of F into F''. If $IT \in C_p(E, F'')$ and either E' or F'' has m.a.p., then $T \in C_p(E, F)$ and $c_p(T) = c_p(IT)$.*

Proof. Assume that E' has m.a.p., and let $S \in J_{p'}(F'', E'') = C_p(E, F'')'$ be a functional which vanishes on the image of $C_p(E, F)$. Then $\langle x', SIy \rangle = 0$ for $x' \in E'$ and $y \in F$, so $SI = 0$. Then also $SIT = 0$, so $\langle T, S \rangle = \text{trace}(S(IT)) = 0$. It follows from the Hahn–Banach theorem that $T \in C_p(E, F)$.

The case in which F'' has m.a.p. is similar. Note that by Grothendieck proposition 40 [12] F has m.a.p. if F'' does. Thus, by Theorem 2.24 $c_p(R) = \gamma_p(R) = \gamma_p(IR) = c_p(IR)$ for each $R \in C_p(E, F)$, that is, we may consider $C_p(E, F) \subseteq C_p(E, F'')$ isometrically. Following the same argument as above, $T''S$ is $\sigma(F'', F')$, $\sigma(F, F')$ continuous from F'' to F, and vanishes on F.

COROLLARY 2.29. *Suppose that E' or F'' has m.a.p. Then $T \in C_p(E, F)$ iff $T' \in C_{p'}(F', E')$, and in either case $c_p(T) = c_{p'}(T')$.*

Proof. Certainly if $T \in C_p(E, F)$ then $T' \in C_{p'}(F', E')$ and $c_{p'}(T') \leqslant c_p(T)$. Conversely, if $T' \in C_{p'}(F', E')$ then $IT = T'' \mid E$ is in $C_p(E, F'')$ and $c_p(IT) \leqslant c_{p'}(T')$. The desired conclusion now follows from the preceeding corollary.

The only nontrivial fact in Table I which now lacks proof is that $(N^p)' = N_p$ with equality of norms. This result follows from the definitions involved and an argument similar to that used in Corollary 2.28.

3. SOME APPLICATIONS TO $\mathscr{L}_p$-SPACES

We now apply some of the previous results to obtain several theorems concerning the $\mathscr{L}_p$-spaces of Lindenstrauss and Pelczynski [22]. In particular we prove an omnibus theorem which includes results of Cohen [4], Holub [14], Johnson [15], Kwapien [18, 19] Lewis [21] and Persson [27], as well as some new results. All results proved below are valid for $\mathscr{L}_p$-spaces.

Our first proposition is routine and so the proof is omitted.

PROPOSITION 3.1. *If* $[A, \alpha]$ *and* $[B, \beta]$ *are Banach ideals and* E *and* F *are such that* $[A(E,F), \alpha] \subseteq [B(E,F), \beta]$ *and* $\alpha(T) \leqslant \beta(T)$ *for every* $T \in B(E,F)$, *then* $[A(E,F), \beta]$ *is a Banach ideal and* β *is equivalent to* α. *Hence,* $[A^{\Delta}(F, E), \alpha^{\Delta}] = [B^{\Delta}(F, E), \beta^{\Delta}]$ *in this case.*

Proposition 3.1 allows us to work with inclusions instead of equality in our various applications.

We first give short proofs of results of Cohen [3] and Persson [27] which will be used in the proof of the main theorem.

THEOREM 3.2. *For any Banach space* G,

$$\Gamma_p{}^*(L_p, G) = J_{p'}(L_p, G) = I_1(L_p, G).$$

Proof. Clearly $\mathscr{L}(G, L_p) = \Gamma_p(G, L_p)$ with equality of norms, hence, $\mathscr{L}^{\Delta}(L_p, G) = \Gamma_p{}^{\Delta}(L_p, G)$, and the result follows by Theorem 2.5.

We recall that Kwapien's representation theorem for $\Gamma_p{}^*(L_p, G)$ says that $T \in \Gamma_p{}^*(L_p, G)$ if and only if $T = UV$ where V is p'-absolutely summing and U' is p-absolutely summing. We remark that the above theorem is false if the role of the operators U and V are interchanged. Indeed, let $T \in \Pi_1(l_2, l_2) \backslash I_1(l_2, l_2)$ (any nonnuclear Hilbert–Schmidt operator suffices). Then T factors

$$\begin{tikzcd} l_2 \arrow[rr, "T"] \arrow[dr, "V"'] & & l_2 \\ & C(K) \arrow[ur, "U"'] & \end{tikzcd}$$

But $V \in D_2(l_2, C(K))$ [22] and $U \in \Pi_2(C(K), L_2)$ [22] but $UV = T$ is not integral.

We now give a new proof of a result of Persson [27].

THEOREM 3.3. *Let* E *be a Banach space. If* $T \in \Pi_{p'}(L_p, E)$, *then* $T' \in I_{p'}(E', L_{p'})$ *and* $\pi_{p'}(T) \geqslant i_{p'}(T')$.

Proof. Let $S \in (I_{p'}')^*\,(E, G) = (I_{p'}^*)'\,(E, G) = \Pi_p'(E, G)$. If $T \in \Pi_{p'}(L_p, E)$, $ST \in J_{p'}(L_p, G) = I_1(L_p, G)$ by the above theorem. Thus, by Corollary 2.21 $T \in I_{p'}'(L_p, E)$ or $T' \in I_{p'}(E', L_{p'})$.

As an immediate corollary we obtain a result analogous to Theorem 3.2.

COROLLARY 3.4. *If $r \geqslant q$ and $1/s = 1/r + 1/q'$, then $\Gamma_r{}^*(L_q, E) \subset I_s(L_q, E)$ and, thus, (taking conjugates) $\Pi_{s'}(E, L_q) \subset \Gamma_r(E, L_q)$.*

We now give an alternative proof of a result which was first proved in [34].

THEOREM 3.5. *Let E be a Banach space. Then E is a $\mathscr{L}_\infty$-space if and only if $\Pi_1(E, F) = I_1(E, F)$ for all F.*

Proof. Suppose first that $\Pi_1(E, F) = I_1(E, F)$ for all F. Thus, for any finite dimensional F we have $I_\infty(F, E) = \Pi_1{}^\Delta(F, E) = I_1{}^\Delta(F, E) = \mathscr{L}(F, E)$ with a uniform bound comparing the equivalent norms and so E'' is injective [23]. Thus, by [23], E is an $\mathscr{L}_\infty$-space.

For the converse, if E is an $\mathscr{L}_\infty$-space, then E'' is complemented in $L_\infty(\mu)$ for some μ [22], and if $T \in \Pi_1(E, F)$, then $T'' \in \Pi_1(E'', F'')$ and so by Theorem 3.3 $T''' \in I_1(F''', E''')$, and, hence, $T \in I_1(E, F)$. Theorem 3.5 is also true if the roles of E and F are interchanged.

We can now give our main result.

THEOREM 3.6. *Let $1 \leqslant p \leqslant \infty$. The following statements about the Banach space E are equivalent:*

(a) *$I \in \Gamma_p(E, E)$, I the identity on E;*

(b) *$\Gamma_p(F, E) \supseteq C(F, E)$ for all F;*

(c) *$\Gamma_{p'}(E', F') \supseteq C(E', F')$ for all F;*

(d) *$\Gamma_p(E, E) \supseteq C(E, E)$ and E has m.a.p.;*

(e) *$\Gamma_p{}^*(E, F) = I_1(E, F)$ for all F;*

(f) *$\Gamma_p{}^*(F, E) = I_1(F, E)$ for all F;*

(g) *$\Gamma_p{}^*(E, E) = I_1(E, E)$ and E has m.a.p.;*

(h) *for every Banach space G, and every adjoint operator $W' \in \Pi_p(E', G')$ one has $W \in I_p(G, E)$;*

(i) *If $V \in \Pi_{p'}(E, G)$, then $V' \in I_{p'}(G', E')$; and*

(j) *$\Gamma_p(E, F) \supset C(E, F)$ for all F.*

We give the proof first for $1 < p < \infty$ and then discuss the necessary modifications for $p = 1$ or ∞. For the proof of (b) $\Rightarrow$ (a) observe that $C(F, E) \subset \Gamma_p(F, E)$ implies that there is a constant C such that $\gamma_p(T) \leqslant C \| T \|$ for all F with m.a.p. and all $T \in C(F, E)$. For, if this is not the case there is a sequence of Banach spaces (F_n), each with m.a.p., and $T_n \in C(F_n, E)$ such that $\| T_n \| = 1$ and $\gamma_p(T_n) \geqslant n^3$. Let $F = \sum_1^\infty \oplus F_n$ (with c_0-norm) and define $T: F \rightarrow E$

by $T(f_n) = \sum_1^\infty T_n f_n/n^2$, and let $Q_n: F_n \to F$ be the natural inclusion map. Then $T \in C(F, E)$ and so to $\Gamma_p(F, E)$. Thus, T factors:

$$\begin{tikzcd} F \arrow[r, "T"] \arrow[dr, "a"'] & E \arrow[r, "i"] & E'' \\ & L_p \arrow[ur, "b"'] & \end{tikzcd}$$

with $\| b \| \| a \| \leqslant \gamma_p(T) + \epsilon$. But $TQ_n = T_n/n^2$ and so $\gamma_p(T) \geqslant \gamma_p(TQ_n) \geqslant n$, i.e., $\gamma_p(T) = +\infty$, a contradiction.

Thus, let $F = l_1(\Gamma)$ and $T: l_1(\Gamma) \to E$ a surjection. From the table we obtain $\Gamma_p(l_1(\Gamma), E) \supseteq \Gamma_p^{\Delta\Delta}(l_1(\Gamma), E) \supseteq C^{\Delta\Delta}(l_1(\Gamma), E) = \mathscr{L}(l_1(\Gamma), E)$. Thus, $T \in \Gamma_p(l_1(\Gamma), E)$ and since $1 < p < \infty$, E is reflexive. Order the finite dimensional subspaces F of E by $F_1 \leqslant F_2$ if and only if $F_1 \subset F_2$ and let $i_F: F \to E$ be the injection map. By the above argument we obtain a $C > 0$ such that $\gamma_p(i_F) = \gamma_p(Ii_F) \leqslant C \| i_F \| = C$ and it follows that I factors through L_p.

(b) $\Rightarrow$ (c): As we saw above, (b) implies E is reflexive. Thus, if $T \in C(E', F')$ then $T' \in C(F'', E'') = C(F'', E)$ and so by (b) $T' \in \Gamma_p(F'', E)$. Thus, $T'' \in \Gamma_{p'}(E', F''')$ and since T'' maps into F' $T \in \Gamma_{p'}(E', F')$.

(c) $\Rightarrow$ (b) follows in the same fashion.

(d) $\Rightarrow$ (a): Since E has m.a.p. we have $\Gamma_p(E, E) = \Gamma_p^{**}(E, E) \supseteq \Gamma_p^{\Delta\Delta}(E, E) \supseteq C^{\Delta\Delta}(E, E) = \mathscr{L}(E, E)$.

(a) $\Rightarrow$ (*d*) follows since any $L_p(\mu)$ has m.a.p.

(a) $\Rightarrow$ (e): Now (a) says that $\mathscr{L}(F, E) = \Gamma_p(F, E)$ for all F and so $I_1(E, F) = \mathscr{L}^\Delta(E, F) = \Gamma_p{}^\Delta(E, F) \subseteq \Gamma_p{}^*(E, F) \subseteq I_1(E, F)$ (the last inclusion follows from Theorem 3.2). Thus, $\Gamma_p{}^*(E, F) = I_1(E, F)$.

(e) $\Rightarrow$ (a): For any F with m.a.p. we get $\Gamma_p(F, E) = \Gamma_p^{**}(F, E) \supseteq \Gamma_p^{*\Delta}(F, E) = I_1{}^\Delta(F, E) = \mathscr{L}(F, E)$. Letting F vary over the finite dimensional subspaces of E yields (a).

(a) $\Rightarrow$ (h): We have by (a) that E is reflexive and so we obtain the factorization

$$\begin{tikzcd} E \arrow[rr, "I"] \arrow[dr, "U"'] & & E \\ & L_p \arrow[ur, "V"'] & \end{tikzcd}$$

If $W' \in \Pi_p(E', G')$, $W'U' \in \Pi_p(L_{p'}, G')$ and so by Theorem 3.3, $UW \in I_p(G, L_p)$ and, thus, $VUW = W \in I_p(G, E)$.

(h) $\Rightarrow$ (f): Let $U \in \Gamma_p{}^*(F, E)$. By Theorem 2.15 there exists G, $V \in \Pi_{p'}(F, G)$ and $W' \in \Pi_p(E', G')$ such that $U = WV$. By (h) $W \in I_p(G, E)$ and so by Lemma 2.3 $U = WV \in I_1(F, E)$, i.e., $\Gamma_p{}^*(F, E) \subseteq I_1(F, E)$ and so by Theorem 3.2 we have equality.

(f) $\Rightarrow$ (a): For all F with m.a.p. we obtain $\Gamma_p(E, F) \supseteq \Gamma_p^{*\Delta}(E, F) = I_1{}^\Delta(E, F) = \mathscr{L}(E, F)$ and it follows that $\mathscr{L}(F', E') = \Gamma_{p'}(F', E')$ for all F'' with m.a.p. It follows as before that the identity on E' factors through $L_{p'}$ and so the identity on E factors through L_p. The proof of the equivalence (a) $\Leftrightarrow$ (g) is similar to that of (a) $\Leftrightarrow$ (d).

(a) $\Rightarrow$ (j) is obvious.

(j) $\Rightarrow$ (a): (j) implies that $\Gamma_{p'}(F', E') \supseteq C(F', E')$ for all F' which, as before, implies (a).

(a) $\Rightarrow$ (i): This implication is the same as the proof of Theorem 2.23.

(i) $\Rightarrow$ (a): (i) implies $I \in \Gamma_p^{**}(E, E) = \Gamma_p(E, E)$ since γ_p is perfect.

Most of the implications for $p = 1$ or ∞ are proved in much the same fashion as above. We indicate a few of the differences.

(a) $\Rightarrow$ (c) for $p = \infty$. Let $T \in C(E', F')$. Then by (a) (since E'' is complemented in E'''') I'' factors

$$\begin{tikzcd} E'' \arrow[rr, "I''"] \arrow[dr] & & E'' \\ & L_\infty \arrow[ur] & \end{tikzcd}$$

and so I' factors through L_1 and so $T \in \Gamma_1(E', F')$. For $p = 1$ the implication (a) $\Rightarrow$ (c) is analogous.

The equivalence (a) $\Leftrightarrow$ (h) for $p = \infty$ is as follows: If $I \in \Gamma_\infty(E, E)$ then E is an $\mathscr{L}_\infty$-space and so has m.a.p. Also by Theorem 3.5 $I_1(G, E) = \Pi_1(G, E)$. Thus, $\mathscr{L}(E, G) = I_1{}^\Delta(E, G) = \Pi_1{}^\Delta(E, G) = I_\infty(E, G)$ which is (h) when $p = \infty$. The implication (h) $\Rightarrow$ (a) is trivial since $I_\infty(E, E) = \Gamma_\infty(E, E)$.

For $p = 1$, if $I \in \Gamma_1(E, E)$ then E' is an $\mathscr{L}_\infty$-space and so $\Pi_1(E', G') = I_1(E', G')$ for any G'. Thus, by a result of Grothendieck [12] if $T' \in \Pi_1(E', G') = I_1(E', G')$, $T \in I_1(G, E)$. On the other hand, if $W' \in \Pi_1(E', G')$ implies $W \in I_1(G, E)$ we have $\Pi_1{}'(G, E) = I_1(G, E)$ and since E has m.a.p.

$$\begin{aligned} \Gamma_1(E, G) &= I_{11}(E, G) = I_\infty{}'(E, G) = \Pi_1^{\Delta\prime}(E, G) \\ &= \Pi_1'^{\Delta}(E, G) = I_1{}^\Delta(E, G) = \mathscr{L}(E, G) \end{aligned}$$

for any G with m.a.p. Thus, $\Gamma_1{}^*(G, E) = I_1(G, E)$ which is (f) for G with m.a.p. However, this was all that was needed to prove (f) $\Rightarrow$ (a).

The equivalence (a) $\Leftrightarrow$ (i) uses Theorem 2.23. The other implications are proved in essentially the same manner as above, using the result (a) $\Rightarrow$ (c) to avoid the use of reflexivity.

We now prove some results which give operator characterizations of finite dimensional spaces. Our initial result of this nature was first given in [24].

THEOREM 3.7. *If E has m.a.p. and $C(E, E) \subset \Pi_p(E, E)$, then E is finite dimensional.*

Proof. From the fact that π_p is perfect and E has m.a.p. we obtain $\mathscr{L}(E, E) = C^{\Delta\Delta}(E, E) \subset \Pi_p^{\Delta\Delta}(E, E) = \Pi_p(E, E)$. Thus, the identity operator on E is p-absolutely summing. By [29] E is finite dimensional.

THEOREM 3.8. *Let E have m.a.p. and $1 < p \leqslant +\infty$. If $I_1(E, E) = I_p(E, E)$ or $N_1(E, E) = N_p(E, E)$ then E is finite dimensional.*

Proof. In either case we obtain $\Pi_{p'}(E, E) = \mathscr{L}(E, E)$ and again it follows that E must be finite dimensional.

We end this section with two results in the spirit of [32]. We remark that there are numerous results of this nature which may be obtained from the preceeding results.

THEOREM 3.9. *Let $\infty \geqslant s \geqslant p \geqslant 1$ and $1/r = 1/s + 1/p'$. Then $\gamma_p{}^*(U) \geqslant i_r(U)$ for all $U \in \mathscr{L}(E, L_s)$.*

Proof. If $U \in \Gamma_p{}^*(E, L_s)$ and $\epsilon > 0$ there exists $W \in \Pi_{p'}(E, G)$ and $V' \in \Pi_p(L_s', G')$ such that $U = VW$ and

$$\begin{aligned}\gamma_p{}^*(U) &\geqslant \pi_{p'}(W)\,\pi_p(V') - \epsilon \geqslant \pi_{p'}(W)\,\pi_s(V') - \epsilon\\ &\geqslant \pi_{p'}(W)\,i_s(V) - \epsilon \geqslant i_r(VW) - \epsilon = i_r(U) - \epsilon.\end{aligned}$$

The above inequalities result from Theorem 3.6(h), the fact that $\pi_a \leqslant \pi_b$ for $b \leqslant a$, and the composition formula of [29].

COROLLARY 3.10. *Let $\infty \geqslant s \geqslant p \geqslant 1$ and $1/r = 1/p' + 1/s$. Then for any $U \in \mathscr{L}(L_s, E)$, $\pi_{r'}(U) \geqslant \gamma_p(U)$.*

The proof is obtained by taking conjugate ideals.

Remark. From Theorem 3.9 we have in addition that for any closed subspace $F \subset L_s(\mu)$, if $I: F \to L_s(\mu)$ is the inclusion map and $T \in \mathscr{L}(E, F)$, then

$$\gamma_p^*(T) \geqslant \gamma_p^*(IT) \geqslant i_r(IT) \geqslant \pi_r(IT) = \pi_r(T),$$

and taking conjugates, $i_{r'}(U) \geqslant \gamma_p(U)$ for all $U \in \mathscr{L}(F, E)$. In particular, taking $s = 1$ and $E = F$, one obtains the result of [45], that the projection constant of F is greater than or equal to the projection constant of F'.

4. Reflexivity of Π_1(E, F)

Clearly if $\Pi_1(E, F)$ is reflexive then both E and F must also be reflexive. In this section the converse will be proved if E has a.p. This answers a question raised by Saphar [37].

First let us recall the definition [13] of $|\ |_{\wedge\backslash}$, the ***greatest right injective*** $\otimes$-norm; given Banach spaces E and F with $F \subset C = C(S)$ isometrically, $E \overset{\wedge\backslash}{\otimes} F$ may be defined as the closure of $E \otimes F$ in $E \hat{\otimes} C$. It is known from [13] that there is a natural norm decreasing map from $E' \overset{\wedge\backslash}{\otimes} F$ into $\Pi_1(E, F)$, and that this natural map is an into isometry whenever E' has m.a.p.

The following lemma is well known in another form [28]; however, it is crucial to the proof of the theorem, so a brief proof is given.

Lemma 4.1. *If E is reflexive and has a.p., then $E' \overset{\wedge\backslash}{\otimes} F = \Pi_1(E, F)$ naturally and isometrically.*

Proof. From [12] E' has m.a.p., so by the remarks above the natural map is an into isometry. To show that the natural map is onto let $U \in \Pi_1(E, F)$ and $F \subset C = C(S)$ isometrically. Then U is integral as an operator into C, so by [12] U is nuclear as an operator into C. By the definition of $|\ |_{\wedge\backslash}$ it is enough to prove that U is in the weak closure of $E' \otimes F$ in $E' \hat{\otimes} C$.

Let $A \in (E' \hat{\otimes} C)' = \mathscr{L}(C, E'')$ be any functional vanishing on $E' \otimes F$. Then $\langle U, A\rangle = \langle(1 \otimes A)(U), tr\rangle$, where tr is the functional on $E' \hat{\otimes} E''$ induced by $\langle\ ,\ \rangle$. If V is the operator from E' to E' defined by $(1 \otimes A)(U)$, then $V = (AU)' \mid E'$. Since $U(E) \subset F$ and $A \mid F = 0$, $V = 0$. By [12], $\langle U, A\rangle = \langle(1 \otimes A)(U), tr\rangle = 0$, since E' has a.p.

Theorem 4.2. *If E and F are reflexive as E had a.p., then $\Pi_1(E, F)$ is reflexive.*

Proof. We will prove that an arbitrary sequence (U_n) in the closed unit ball of $\Pi_1(E,F)$ has a weakly convergent subsequence. First notice that by Lemma 4.1 each U_n is compact, so there is a separable subspace F_0 of F such that $U_n(E) \subset F_0$ for all n. Since $\Pi_1(E,F_0) \subset \Pi_1(E,F)$ isometrically, it is enough to show that some subsequence converges weakly in $\Pi_1(E,F_0)$ to an element of $\Pi_1(E,F_0)$. Thus, in the remainder of the proof we may assume that F itself is separable.

The first claim is that (U_n) has a subsequence which converges in the weak operator topology to an element of $\Pi_1(E,F)$. The space F is separable and reflexive, so F' contains a countable set D whose linear span is norm dense in F'. For $y' \in D$ and each n we have $\| U_n'y' \| \leqslant \pi_1(U_n) \| y' \| \leqslant \| y' \|$. By appealing to the reflexivity of E' and using a diagonalization argument we may assume, by passing to a subsequence if necessary, that $(U_n'y')_{n\geqslant 1}$ is weakly convergent for each $y' \in D$. It follows that for each $x \in E$ the sequence $(U_n x)$ must be weakly Cauchy in F; in fact, given $z' \in F'$ and $\epsilon > 0$, choose y' in the linear span of D so that $\| z' - y' \| < \epsilon$. Then $|\langle U_n x - U_m x, z' \rangle| \leqslant 2 \| x \| \epsilon + |\langle x, U_n'y' - U_m'y' \rangle|$, and the last term converges to zero because $(U_n'y')$ is weakly convergent in E'. The space F is weakly sequentially complete so $Ux = wk - \lim_n U_n x$ exists for each $x \in E$. The function U is linear and is bounded by the Banach–Steinhaus theorem. From the finite series definition of the 1-summing operators it is also clear that U must be 1-summing and that $\pi_1(U) \leqslant \limsup_n \pi_1(U_n) \leqslant 1$.

Therefore, in the remainder of the proof we may suppose, by subtracting the weak operator limit if necessary, that $\pi_1(U_n) \leqslant 1$ and that (U_n) converges to zero in the weak operator topology. It will be shown that this implies that (U_n) converges to zero weakly in $\Pi_1(E,F)$.

Let $A \in \Pi_1(E,F)' = (E' \hat{\otimes} F)'$ and choose a compact Hausdorff S so that there is an isometric embedding J of F into $C = C(S)$. Now $1 \otimes J$ is an isometric embedding of $E' \hat{\otimes} F$ into $E' \hat{\otimes} C$, so by the Hahn–Banach theorem there is a $B \in (E' \hat{\otimes} C)' = \mathscr{L}(E', M(S))$ such that $\| A \| = \| B \|$ and $\langle U, A \rangle = \langle (1 \otimes J)(U), B \rangle$ for all $U \in \Pi_1(E,F)$. The extension of $B \otimes 1$ defines an operator from $E' \hat{\otimes} F$ to $M(S) \hat{\otimes} F$ of norm at most $\| B \|$, and by [12, Corollary 3, p. 61], $M(S) \hat{\otimes} F = M(S) \hat{\otimes} F$ isometrically. Consider $J' \in \mathscr{L}(M(S), F') = (M(S) \hat{\otimes} F)'$. By checking elementary tensors it is easy to see that

$$\langle U, A \rangle = \langle (B \otimes 1)(U), J' \rangle$$

for all $U \in \Pi_1(E,F)$. Each of the tensors $(B \otimes 1)(U_n)$ induces a nuclear operator from F' into $M(S)$, and by [12, Theorem 11, p. 141], each such operator maps the closed unit ball of F' into the set of measures dominated (in the natural order of $M(S)$) by some positive μ_n. Let $\mu = \sum_{n\geqslant 1} 2^{-n} \| \mu_n \|^{-1} \mu_n$. Since $L_1(\mu) \subset M(S)$ is the range of a norm one projection P, $L_1(\mu) \hat{\otimes} F$ is naturally isometrically contained in $M(S) \hat{\otimes} F$ and $P \otimes 1$ is a projection of $M(S) \hat{\otimes} F$ onto $L_1(\mu) \hat{\otimes} F = L_1(\mu, F)$. Let $f_n \in L_1(\mu, F)$ be $(P \otimes 1)(B \otimes 1)(U_n) = [(PB) \otimes 1](U_n)$. Then for each n

$$\langle U_n , A\rangle = \langle f_n , J' \mid L_1(\mu)\rangle$$

and

$$\langle U_n((PB)' \chi_D)), y'\rangle = \langle f_n , \chi_D \otimes y'\rangle$$

for every Borel set $D \subset S$ and $y' \in F'$. By the first equality the proof will be complete if we show that (f_n) converges to zero weakly in $L_1(\mu, F)$. By [2] this will follow if

(a) $\int_D f_n \, d\mu \to 0$ weakly for every Borel set $D \subset S$; and

(b) the sequence $(\| f_n(\cdot)\|) \subset L_1(\mu)$ is uniformly countably additive.

The first condition holds since, by the natural identification $L_1(\mu, F) = L_1(\mu) \hat{\otimes} F$,

$$\left\langle \int_D f_n \, d\mu, y' \right\rangle = \langle f_n , \chi_D \otimes y'\rangle$$
$$= \langle U_n((PB)' (\chi_D)), y'\rangle,$$

and because we know that (U_n) converges to zero in the weak operator topology.

To establish the second condition suppose the contrary. Then there is a $\delta > 0$ and a pairwise disjoint sequence (D_i) of Borel sets such that

$$\sup_n \sum_{i\geqslant m} \int_{D_i} \| f_n \| \, d\mu > \delta$$

for each m. Choose an increasing sequence $(m_k)_{k\geqslant 1}$ and a sequence $(n_k)_{k\geqslant 1}$ so that

$$\sum_{i=m_k}^{m_{k+1}-1} \int_{D_i} \| f_{n_k} \| \, d\mu > \delta$$

for each k. Let λ_k be the Bochner indefinite integral of f_{n_k}, so that the total variation of λ_k over $D \subset S$ is given by $V(\lambda_k , D) = \int_D \| f_{n_k} \| \, d\mu$.

By the previous inequality and the definition of $v(\lambda_k, \cdot)$ there is an increasing sequence $(t_k)_{k\geqslant 1}$ of positive integers and a pairwise disjoint collection $(G_j)_{j\geqslant 1}$ of disjoint Borel sets so that, for each k,

$$\bigcup_{j=t_k}^{t_{k+1}-1} G_j = \bigcup_{i=m_k}^{m_{k+1}-1} D_i$$

and

$$\sum_{j=t_k}^{t_{k+1}-1} \| \lambda_k(G_j)\| > \delta.$$

For $t_k \leqslant j < t_{k+1}$ choose $w_j' \in F'$ satisfying $\| w_j' \| = 1$ and $\| \lambda_k(G_j)\| = |\langle \lambda_k(G_j), w_j' \rangle|$. The series $\sum_j \chi_{G_j}$ is weakly unconditionally Cauchy in $L_\infty(\mu)$ because the G_j's are pairwise disjoint. Thus $\sum_j (PB)'(\chi_{G_j})$ is weakly unconditionally Cauchy in E, and, hence, unconditionally convergent because E is reflexive. Choose k so that

$$\sup_{\| x' \| \leqslant 1} \sum_{j=t_k}^{t_{k+1}-1} |\langle (PB)'(\chi_{G_j}), x' \rangle| < \delta/2.$$

Then summing for $t_k \leqslant j < t_{k+1}$ we have

$$\begin{aligned}
\delta &< \sum_j |\langle \lambda_k(G_j), w_j' \rangle| \\
&= \sum_j \left| \left\langle \int_{G_j} f_{n_k}\, d\mu, w_j' \right\rangle \right| \\
&= \sum_j |\langle U_{n_k}((PB)'(\chi_{G_j})), w_j' \rangle| \\
&\leqslant \sum_j \| U_{n_k}((PB)'(\chi_{G_j}))\| \\
&\leqslant \pi_1(U_{n_k}) \sup_{\| x' \| \leqslant 1} \sum_j |\langle (PB)'(\chi_{G_j}), x' \rangle| \\
&< \delta/2.
\end{aligned}$$

This contradiction completes the proof.

From [13] recall the definition of the *least right projective* $\otimes$-norm $| \ |_{\vee /}$; given Banach spaces E and F, with $Q: L_1(\mu) \to F$ a quotient map, $E \check{\otimes}/ F$ is defined so that $1 \otimes Q$ induces a quotient map from $E \check{\otimes} L_1(\mu)$ onto $E \check{\otimes}/ F$, where $| \ |_\vee$ is the least $\otimes$-norm.

Corollary 4.3. *If E and F are reflexive and E has a.p., then both $E \hat{\otimes}\backslash F$ and $E \check{\otimes}/ F$ are reflexive. Further, $(E \hat{\otimes}\backslash F)' = E' \check{\otimes}/ F'$ and*

$(E \check{\otimes}' F)' = E' \hat{\otimes}^{\backslash} F'$, where both identifications are natural and isometric.

Proof. By [13] and Lemma 4.1 $(E \check{\otimes}' F)' = \Pi_1(E, F') = E' \hat{\otimes}^{\backslash} F'$. By Theorem 4.2 $(E' \hat{\otimes}^{\backslash} F')' = (E \check{\otimes}' F)'' = E \check{\otimes}' F$.

THEOREM 4.4. *Let E, F be reflexive with F having a.p. If $T \in \mathscr{L}(E, F)$ factors through a $C(S)$-space, then T factors compactly through c_0.*

Proof. Define $\varphi \in \Pi_1(F, E)'$ by $\varphi(S) = \operatorname{trace}(TS)$. Since $\Pi_1(F, E)$ is reflexive, $\varphi \in \Pi_1(F, E)' = N_\infty(E, F)'' = N_\infty(E, F)$.

5. Applications to Hilbert Spaces

In this section we study the norms i_{pq} and j_{pq} for operators between Hilbert spaces. In particular we compute the relative projection constants of isometric copies of Hilbert spaces in L_p-spaces.

DEFINITION 5.1. Given a Banach space E and a compact topological group G, a (G, E)-representation is a continuous homomorphism $g \to a_g{}^E$ of G into the group of isometries of E. Say that $T \in \mathscr{L}(E, F)$ is invariant under the (G, E), (G, F)-representation, if $Ta_g{}^E = a_g{}^F T$ for each $g \in G$.

We shall need the following lemma, which is a generalization of Corollary 1 of [11].

LEMMA 5.2. *Let E, F be n-dimensional and $T \in \mathscr{L}(E, F)$ be invertible. Suppose $S \in \mathscr{L}(E, F)$ is invariant under the (G, E), (G, F)-representations iff $S = \lambda T$ for some scalar λ. Then for every ideal norm α, $\alpha^\Delta(T)\, \alpha(T^{-1}) = n$.*

Proof. Let $L \in \mathscr{L}(F, E)$ be such that $\alpha(L) = 1$ and $\alpha^\Delta(T) = \operatorname{trace}(LT)$. Let dg be the normalized Haar measure on G, and define $L_0 \in \mathscr{L}(F, E)$ by

$$L_0 = \int_G a_{g^{-1}}^E L a_g{}^F \, dg.$$

By the invariance of T we have

$$L_0 T = \int_G a_{g^{-1}}^E L T a_g{}^E \, dg$$

and by the translation invariance of dg

$$L_0 T a_h{}^E = a_h{}^E L_0 T \qquad \text{for each} \quad h \in G.$$

Then $TL_0Ta_h^E = Ta_h^EL_0T = a_h^FTL_0T$, and hence, $TL_0T = \lambda T$ for some λ, so that finally $L_0 = \lambda T^{-1}$. We obtain

$$\alpha^\Delta(T) = \operatorname{trace}(LT) = \operatorname{trace}(L_0T) = \lambda n$$

and $\lambda\alpha(T^{-1}) = \alpha(L_0) \leqslant \alpha(L) = 1$; therefore, $\alpha^\Delta(T)\,\alpha(T^{-1}) \leqslant n$. But clearly $\alpha^\Delta(T)\,\alpha(T^{-1}) \geqslant \operatorname{trace}(TT^{-1}) = n$.

COROLLARY 5.3. *For each injection map $I\colon l_p^n \to l_q^n$, $\alpha^\Delta(I)\,\alpha(I^{-1}) = n$.*

Proof. Since the group of isometries formed by the permutations and changes of signs on the coordinates of the vectors in l_p^n and l_q^n, respectively, satisfy the above conditions when $T = I$.

Remark. The above corollary simplifies the calculation in [32, Sections 7 and 8] regarding the norms $\alpha = \pi_p$ and $\alpha^\Delta = \nu_{p'}$.

LEMMA 5.4. *Let E, F be finite dimensional and K_1, K_2 be as in Corollary 2.17. Let G be a compact topological group and $T \in \mathscr{L}(E, F)$ be invariant under the (G, E), (G, F)-representations. Then the measures μ and ν of Corollary 2.17 can be also taken to satisfy: $\mu(f) = \mu(f \circ (a_g^E)')$ for every $f \in C(K_1)$, $\nu(h) = \nu(h \circ a_g^E)$ for every $h \in C(K_2)$, for all $g \in G$.*

Proof. For simplicity write $a_g^E = a_g$ and $a_g^F = b_g$, and let μ, ν and $K > 0$ satisfy

$$|\langle Tx, z\rangle| \leqslant K\mu(|f_x|^p)^{1/p}\,\nu(|f_z|^{q'})^{1/q'}$$

for every $x \in E$ and $z \in F'$, where $f_x \in C(K_1)$ and $f_z \in C(K_2)$ are the functions naturally defined by x and z. Then for each $g \in G$

$$\begin{aligned}|\langle Tx, z\rangle| &= |\langle Ta_gx, (b_g^{-1} - 1)'z\rangle| \\ &\leqslant K\mu(|f_x \circ a_g'|^p)^{1/p}\,\nu(|f_z \circ b_g^{-1}|^{q'})^{1/q'}.\end{aligned}$$

For each $t > 0$ we have by Lemma 2.7 that

$$K^{-r}r^{-1}\,|\langle Tx, z\rangle|^r \leqslant p^{-1}t^p\mu(|f_x \circ a_g'|^p) + (q')^{-1}\,t^{-q'}\nu(|f_z \circ b_g^{-1}|^{q'}),$$

where $r^{-1} = p^{-1} + (q')^{-1}$. Let dg be normalized Haar measure on G and define $\hat{\mu}$ and $\hat{\nu}$ by

$$\hat{\mu}(f) = \int_G \mu(f \circ a_g')\,dg, \qquad f \in C(K_1),$$

$$\hat{\nu}(f) = \int_G \nu(f \circ b_g^{-1})\,dg, \qquad f \in C(K_2).$$

Clearly $\hat{\mu}$ and $\hat{\nu}$ are normalized and invariant. Integrating the last inequality over G and applying Lemma 2.7 gives

$$|\langle Tx, z\rangle| \leqslant K\hat{\mu}(|f_x|^p)^{1/p}\,\hat{\nu}(|f_z|^{q'})^{1/q'}.$$

This establishes the lemma.

For α an ideal norm and E a Banach space, $\alpha(E)$ means the α-norm of the identity operator on E. Of course, it may happen that $\alpha(E) = \infty$ for E infinite dimensional.

THEOREM 5.5. *If* $1 \leqslant q \leqslant p \leqslant \infty$ *then* $j_{pq}(l_2^n) = \pi_p(l_2^n)\,\pi_{q'}(l_2^n)$.

Proof. The inequality $j_{pq}(l_2^n) \leqslant \pi_p(l_2^n)\,\pi_{q'}(l_2^n)$ follows immediately from the typical factorization of the J_{pq}-operators. For the other inequality, let μ and ν be normalized measures on S_n (the boundary of the unit sphere of l_2^n) such that, for $x, y \in l_2^n$,

$$|\langle x, y\rangle| \leqslant j_{pq}(l_2^n)\,\mu(|\langle x, \cdot\rangle|^p)^{1/p}\,\nu(|\langle y, \cdot\rangle|^{q'})^{1/q'}.$$

By Lemma 5.4 both μ and ν may be assumed to be invariant under isometries of l_2^n, so $\mu = \nu = m$, the unique normalized rotational invariant measure on S_n. By [10] $m(|\langle z, \cdot\rangle|^s)^{1/s} = \|z\|\,\pi_s(l_2^n)^{-1}$ for all $z \in l_2^n$ and $1 \leqslant s < \infty$, so that maximizing the above inequality for $\|x\| = 1$, $\|y\| = 1$ gives

$$1 \leqslant j_{pq}(l_2^n)\,\pi_p(l_2^n)^{-1}\,\pi_{q'}(l_2^n)^{-1}.$$

COROLLARY 5.6. *For* $1 \leqslant q \leqslant p \leqslant \infty$, $i_{pq}(l_2^n)\,\pi_p(l_2^n)\,\pi_{q'}(l_2^n) = n$.

Proof. By Corollary 1 of [11] $n = i_{pq}(l_2^n)\,i_{pq}^{\Delta}(l_2^n) = i_{pq}(l_2^n)\,j_{q'p'}(l_2^n)$.

COROLLARY 5.7. *If* $1 < q \leqslant p < \infty$ *and H is an infinite dimensional Hilbert space, then*

$$i_{pq}^{**}(H) = 2\Pi^{-1/2p-1/2q'}\Gamma\left(\frac{p+1}{2}\right)^{1/p}\Gamma\left(\frac{q'+1}{2}\right)^{1/q'}.$$

Proof. The norm i_{pq}^{**} is perfect so $i_{pq}^{**}(H) = \sup_n i_{pq}(l_2^n) = \lim_n i_{pq}(l_2^n)$, the last since $i_{pq}(l_2^n) \leqslant i_{pq}(l^{n+1})$ for each n. The limit may be calculated using the preceeding corollary, the expressions given in [10] for $\pi_p(l_2^n)$ and $\pi_{q'}(l_2^n)$, and Stirling's formula.

Remarks. (1) It is clear that $i_{pq}(l_2^n) \leqslant i_{pq}(H)$ for all n, so $i_{pq}^{**}(H) \leqslant i_{pq}(H)$. We do not know if it is always the case that $i_{pq}^{**}(H) = i_{pq}(H)$. However, the equality is true when $1 < p = q < \infty$, since then $i_{pq} = \gamma_p$ is perfect.

(2) For $1 < q \leqslant p < \infty$, $i_{pq}(l_2) < \infty$; in fact the span of the Radamacher system in both $L_p[0, 1]$ and $L_q[0, 1]$ is isomorphic to l_2. It is well known (cf. [22]) that $\gamma_p(H) < \infty$ for any Hilbert space H and $1 < p < \infty$.

Grothendieck observed the following result [12].

THEOREM 5.8. *Let H_1 and H_2 be Hilbert spaces, and $T \in \mathscr{L}(H_1, H_2)$. Then $T \in N_1(H_1, H_2)$ if and only if T is compact and the sequence of eigenvalues $\{\lambda_i\}$ of the Hermitian operator $U = (T^*T)^{1/2}$ is absolutely summable. Moreover, $\nu_1(T) = \nu_1(U) = \sum |\lambda_i|$.*

For any ideal norm α, and for any $T \in \mathscr{L}(H_1, H_2)$ we have $\alpha(T) = \alpha((T^*T)^{1/2})$, since T is obtained from $(T^*T)^{1/2}$ by partial isometry. For this reason we shall consider now only diagonal nonnegative multiplication operators on a Hilbert space H.

THEOREM 5.9. *Let $T \in \mathscr{L}(l_2^n, l_2^n)$ be diagonal multiplication by a nonnegative sequence $(\lambda_i)_1^n$. Then for all $1 \leqslant q \leqslant p \leqslant \infty$*

$$j_{pq}(T) \leqslant \sum_1^n \lambda_i \leqslant i_{pq}(l_2^n)\, j_{pq}(T).$$

Further,

$$i_{pq}(l_2^n)\, j_{pq}(T) \leqslant n \left(\int_{S_n} \left(\sum \lambda_i x_i^2\right)^{p/2} dm(x)\right)^{1/p} \left(\int_{S_n} \left(\sum \lambda_i x_i^2\right)^{q'/2} dm(x)\right)^{1/q'},$$

where dm is the normalized rotational invariant measure on the sphere S_n, with equality when T is the identity.

Proof. Let I be the identity on l_2^n, then by Lemma 2.3 we have

$$i_1(T) = i_1(TI) \leqslant j_{pq}(T)\, j_{pq}^{\Delta}(I) = j_{pq}(T)\, i_{pq}(l_2^n),$$

and by Theorem 5.8

$$j_{pq}(T) \leqslant \nu_1(T) = \sum \lambda_i .$$

Let U be diagonal multiplication by $(\lambda_i^{1/2})_{i=1}^n$. Then $j_{pq}(T) \leqslant \pi_p(U)\, \pi_{q'}(U)$, since U is self-adjoint. Let

$$c_t = \left(\int_{S_n} \| Ua \|^t \, dm(a)\right)^{1/t}, \qquad 1 \leqslant t < \infty,$$

and define now the probability measure μ on S_n by

$$\mu(f) = c_t^{-t} \int_{S_n} f(\| Ua \|^{-1} Ua) \, \| Ua \|^t \, dm(a),$$

$f \in C(S_n)$. Then by [10]

$$\mu(|\langle x, \cdot \rangle|^t) = c_t^{-t} \int_{S_n} |\langle x, Ua\rangle|^t \, dm(a)$$

$$= c_t^{-t} \| Ux \|^t (\pi_t(l_2{}^n))^{-t},$$

which implies $\pi_t(U) \leqslant c_t \pi_t(l_2{}^n)$ (see also [8] for this inequality). Combining the inequalities and Corollary 5.6 we obtain the required result.

Remark. Since in [8] it was shown that $\pi_t(U) = c_t \pi_t(l_2{}^n)$ and that this is equivalent to the Hilbert–Schmidt norm $\sigma(U)$, it follows that there is a constant a_{pq} for $1 < q \leqslant p < \infty$, such that for all n and U, $nc_p c_q \leqslant a_{pq} \sum_1^n \lambda_i$.

THEOREM 5.10. *Let* $1 \leqslant q \leqslant p \leqslant \infty$ *and* $T \in \mathscr{L}(E, F)$, *where either* E *or* F *is a Hilbert space* H. *Then*

$$j_{pq}(T) \leqslant i_1(T) \leqslant i_{pq}^{**}(H) \, j_{pq}(T)$$

and

$$j_{pq}^{\Delta}(T) \leqslant i_{pq}^{**}(H) \, \| T \|.$$

Furthermore, if $E = F = H$, *then for compact* T

$$i_{pq}^{**}(H) \, j_{pq}(T) \leqslant \lim_n \left[n \left(\int_{S_n} \left(\sum_{i \leqslant n} \lambda_i x_i{}^2 \right)^{p/2} dm(x) \right)^{1/p} \left(\int_{S_n} \left(\sum_{i \leqslant n} \lambda_i x_i{}^2 \right)^{q'/2} dm(x) \right)^{1/q'} \right],$$

where $(\lambda_i)_{i \geqslant 1}$ *is the sequence of eigenvalues of* $(T^*T)^{1/2}$ *counted according to their multiplicities.*

Proof. For the first inequality, it is sufficient by duality to suppose that $E = H$. Since $i_1(T)$ is the supremum of $i_1(T \mid G)$, $G \subset H$ a finite dimensional subspace, we may suppose that $E = l_2{}^n$. But then by Lemma 2.3, $i_1(T) \leqslant j_{pq}^{\Delta}(l_2{}^n) \, j_{pq}(T) = i_{pq}(l_2{}^n) \, j_{pq}(T) \leqslant i_{pq}^{**}(H) \, j_{pq}(T)$.

For $0 < \epsilon < 1$, choose $S \in \mathscr{F}(F, E)$ with $(1 - \epsilon) \, j_{pq}^{\Delta}(T) \leqslant |\operatorname{trace}(TS)|$ and $1 \geqslant j_{pq}(S)$. Then $(1 - \epsilon) \, j_{pq}^{\Delta}(T) \leqslant |\operatorname{trace}(TS)| \leqslant i_1{}^{\Delta}(T) \, i_1(S) \leqslant \| T \| \, i_{pq}^{**}(H) \, j_{pq}(S) = i_{pq}^{**}(H) \, \| T \|$.

Finally, if $T \in K(H, H)$ the operator T can be written in the form $\sum_{i \geqslant 1} \lambda_i e_i \otimes u_i$, where $\{e_i\}$, $\{u_i\}$ are orthonormal sequences. Let $V = \sum_{i \geqslant 1} \lambda^{1/2} e_i \otimes u_i$, and $V_n = \sum_{i \leqslant n} \lambda^{1/2} e_i \otimes u_i$, and let W_n be V_n regarded as an operator from $[e_i]_1^n$ to $[u_i]_1^n$. Then for every ideal norm α, $\alpha(V_n \mid [e_i]_1^n) = \alpha(W_n)$, since $[u_i]_1^n$ is the range of a norm one projection. Then for all $1 \leqslant t < \infty$,

$$\pi_t(V) = \lim \pi_t(V_n \mid [e_i]_1^n) = \lim \pi_t(W_n)$$

$$= \lim \pi_t(l_2{}^n) \left(\int_{S_n} \left(\sum \lambda_i a_i{}^2\right)^{t/2} dm(a)\right)^{1/t}.$$

Since $V^*V = T$,

$$j_{pq}(T) \leqslant \pi_p(V)\, \pi_{q'}(V) = \lim \pi_p(W_n)\, \pi_{q'}(W_n)$$

and the result follows by Corollaries 5.6 and 5.7.

COROLLARY 5.11. *Let $1 \leqslant q \leqslant p \leqslant \infty$. Then*

$$j_{pq}^{\Delta} \leqslant i_{pq}^{**}(l_2)\, \gamma_2\,,$$

and

$$j_2 \leqslant i_{pq}^{**}(l_2)\, j_{pq}.$$

Proof. Observe that $i_{pq}^{**}(l_2) = i_{pq}^{**}(H)$ for any infinite dimensional Hilbert space H. The first inequality follows immediately from the second inequality of Theorem 5.10. For the other let $T \in J_{pq}(E, F)$ and $S \in \Gamma_2(F, G)$. Factor $S = BA$, where $\| A \|\, \| B \| \leqslant (1 + \epsilon)\, \gamma_2(S)$, $A \in \mathscr{L}(F, H)$, $B \in \mathscr{L}(H, G)$ and, of course, H is a Hilbert space. Then $i_1(ST) \leqslant \| B \|\, i_1(AT) \leqslant (1 + \epsilon)\, \gamma_2(S)\, i_{pq}^{**}(l_2)\, j_{pq}(T)$ by Theorem 5.10. By Proposition 2.20 $T \in \Gamma_2{}^*(E, F) = J_2(E, F)$, and $j_2(T) \leqslant i_{pq}^{**}(l_2)\, j_{pq}(T)$.

Remark. Observe that the inequalities given above are exact for all $1 \leqslant q \leqslant p \leqslant \infty$.

We shall now show that $\gamma_p(H) = i_{pp}^{**}(H)$ can be considered as the relative projection constant of the space H embedded isometrically in the L_p-spaces.

THEOREM 5.12. *Let $1 \leqslant p \leqslant \infty$ and n be a positive integer. There is a measure μ and a subspace $H \subset L_p(\mu)$ isometric to $l_2{}^n$ onto which there is a projection of norm $\gamma_p(l_2{}^n)$.*

Proof. Let $S \subset l_2{}^n$ be the surface of the unit sphere, m rotational invariant measure on S and define T from $l_2{}^n$ into $L_p(m)$ by

$Tx = \pi_p(l_2^n)\langle x, \cdot\rangle$. Clearly T is an isometric embedding. Define Q from $L_p(m)$ to l_2^n by

$$Q(f) = n\pi_p(l_2^n)^{-1} \left(\int_S f(y)\, y_i m(dy)\right)_{i\leqslant n}.$$

Then QT is the identity on l_2^n so TQ is a projection onto $H = T(l_2^n)$. For $f \in L_p(m)$ and $a = (a_i)_{i\leqslant n} \in l_2^n$

$$\begin{aligned} n^{-1}\pi_p(l_2^n)\, &|\langle Q(f), a\rangle| \\ &= \left|\int_S f(y) \left(\sum_{i\leqslant n} a_i y_i\right) m(dy)\right| \\ &\leqslant \| f \|_p \left(\int_S |\langle a, y\rangle|^{p'}\, m(dy)\right)^{1/p'} \\ &= \| f \|_p\, \pi_{p'}(l_2^n)^{-1} \| a \|. \end{aligned}$$

Thus, $\| Q \| \leqslant n\pi_p(l_2^n)^{-1}\, \pi_{p'}(l_2^n)^{-1} = \gamma_p(l_2^n)$ by Corollary 5.6. But $\gamma_p(l_2^n) \leqslant \| T \|\, \| Q \| = \| Q \|$, so that $\| Q \| = \gamma_p(l_2^n)$ and, thus, $\| QT \| = \gamma_p(l_2^n)$.

COROLLARY 5.13. *Let $1 < p < \infty$ and H be an infinite dimensional Hilbert space. There is a measure μ and a subspace of $L_p(\mu)$ which is isometric to H and onto which there is a projection of norm $\gamma_p(H)$.*

Proof. Let $S \subset H'$ be the closed unit ball, $B(S)$ the bounded functions on S, (e_i, e_i') the unit vector basis of l_p and $\mathscr{D}$ the collection of finite dimensional subspaces of H, directed by inclusion. For each $E \in \mathscr{D}$, choose $U_E \in \mathscr{L}(E, l_p^m)$ and $V_E \in \mathscr{L}(l_p^m, E)$ to satisfy

(a) $(1 - (\dim E)^{-1}) \| x \| \leqslant \| U_E x \| \leqslant \| x \|$ for $x \in E$; and

(b) $V_E U_E$ is the identity on E and $\| V_E \| = \gamma_p(E) \leqslant \gamma_p(H)$.

Let $x_i' \in S$, $i = 1, 2,..., m$, be a Hahn–Banach extension of $U_E' e_i'$, define

$$A_E(f) = \left(\sum_{i\leqslant n} | f(x_i')|^p\right)^{1/p}, \qquad f \in B(S)$$

$$B_E(f, y') = \left\langle \sum_{i\leqslant n} f(x_i')\, e_i\, , V_E' y'\right\rangle, \qquad f \in B(S), \qquad y' \in H$$

and

$$\Phi_E(f, y') = (A_E(f), B_E(f, y')).$$

Clearly A_E is a semi-norm on $B(S)$ which satisfies

(c) $(1 - (\dim E)^{-1}) \| x \| \leqslant A_E(f_x) \leqslant \| x \|$ for $x \in E$, and B_E is a bilinear form on $B(S) \times H'$ satisfying

(d) $| B_E(f, y')| \leqslant \gamma_p(H) A_E(f) \| y' \|, f \in B(S), y' \in H'$; and

(e) $B_E(f_x, y') = \langle x, y' \rangle$ for $x \in E, y' \in H'$.

As in Theorem 2.10 the net $\{\Phi_E\}_{E \in \mathscr{D}}$ clusters simply to a function $\Phi(f, y') = (A(f), B(f, y')$. As before, setting $\|\tilde{f}\| = A(f)$ defines a norm on

$$M = \{f \in B(S) : A(f) < \infty\}/\{f \in B(S) : A(f) = 0\}$$

under which the completion of M is isometric to an $L_p(\mu)$-space. Defining U from H to $L_p(\mu)$ by $Ux = \tilde{f}_x$ gives an isometric embedding and defining V from $L_p(\mu)$ to $H'' = H$ so that $\langle y', V(\tilde{f}) \rangle = B(f, y')$ gives a linear operator of norm at most $\gamma_p(H)$ such that VU is the identity on H. Thus, UV is a projection of $L_p(\mu)$ onto $U(H)$ of norm exactly $\gamma_p(H)$.

COROLLARY 5.14. *Let $1 < p < \infty$ and ν be any measure which is not purely atomic. Then $L_p(\nu)$ has a subspace isometric to l_2 (resp. l_2^n) onto which there is a projection of norm $\gamma_p(l_2)$ (resp., $\gamma_p(l_2^n)$).*

Proof. It is known [44] that (a) For ν not purely atomic, $L_p(\nu)$ has a subspace isometric to $L_p[0, 1]$ onto which there is a norm one projection; and (b) If $L_p(\sigma)$ is isometric to a subspace of $L_p(\mu)$, there is a norm one projection onto the subspace.

By Corollary 5.13 $l_2 \subset L_p(\mu)$ for some μ in such a way that l_2 is the image of a norm $\gamma_p(l_2)$ projection. Let $L \subset L_p(\mu)$ be the sublattice generated by l_2. L is separable since l_2 is, L is isometric to a space $L_p(\sigma)$ and by (b) $l_2 \subset L_p(\sigma)$ is $\gamma_p(l_2)$-complemented. Since $L_p(\sigma)$ is separable it is isometric to a subspace of $L_p[0, 1]$. By (b) l_2 is $\gamma_p(l_2)$-complemented in $L_p[0, 1]$, and part (a) completes the proof.

It is known [46] that for any probability measure space (Ω, Σ, μ), E is a closed subspace of $L_p(\mu)(\infty > p \geqslant 2)$ isomorphic to a Hilbert space if and only if there is a constant C_E such that $\| f \|_p \leqslant C_E \| f \|_2$ for every $f \in E$. An exact lower bound for C_E in the isometric case is given by the following theorem.

THEOREM 5.15. (1) *Let $2 \leqslant p \leqslant \infty$, then*

$$\min_{\mu, H} \max_{f \in H} \| f \|_p / \| f \|_2 = i_{p2}(l_2^n)$$

where the minimum is taken over all subspaces $H \subseteq L_p(\mu)$ isometric to l_2^n and probability measures μ.

(2) *Let $1 \leqslant p \leqslant 2$, then*

$$\min_{\mu,H} \max_{f \in H} \|f\|_2/\|f\|_p = \pi_p(l_2^n)/n^{1/2},$$

where the minimum ranges over all subspaces $H \subset L_p(\mu)$ isometric to l_2^n complemented with $\gamma_p(l_2^n)$ norm (minimal) projections, and probability measures μ.

Proof. (1) Consider the following factorization of the identity on $H \subset L_p(\mu)$

$$H \xrightarrow{j} L_p(\mu) \xrightarrow{I} L_2(\mu) \xrightarrow{P} Ij(H) \xrightarrow{(Ij)^{-1}} H,$$

where I, j are the inclusion operators, P is the orthogonal norm one projection. Since $\pi_2(l_2^n) = n^{1/2}$

$$\begin{aligned} n^{1/2}/\pi_p(l_2^n) = i_{p2}(l_2^n) = i_{p2}(H) &\leqslant \|j\| \, \|(Ij)^{-1} P\| \leqslant \|(Ij)^{-1}\| \\ &= \max_{f \in H} \|f\|_p/\|f\|_2 . \end{aligned}$$

On the other hand, letting T: $l_2^n \to L_p(m)$ be as in Theorem 5.12 the isometry $Tx = \pi_p(l_2^n)\langle x, \cdot\rangle$ and $H = T(l_2^n)$, then the proof is concluded by observing that for all $x \in l_2^n$

$$n^{1/2}/\pi_p(l_2^n) = \| Tx \|_p/\| Tx \|_2 .$$

(2) Let $H \subset L_p(\mu)$ be isometric to l_2^n and $P: L_p(\mu) \to H$ be a projection having norm $\gamma_p(l_2^n)$ (the existence of H admitting such norms is assured by Corollary 5.14 whenever μ is a probability non purely atomic measure). Consider the factorization

$$H \xrightarrow{I^{-1}} L_2 \xrightarrow{I} L_p \xrightarrow{P} H$$

for which

$$\begin{aligned} i_{2p}(l_2^n) = i_{2p}(H) &\leqslant \|I^{-1}\| \, \|P\| \\ &= \gamma_p(l_2^n) \max_{f \in H} \|f\|_2/\|f\|_p ; \end{aligned}$$

therefore, $\max_{f \in H} \|f\|_2/\|f\|_p \geqslant i_{2p}(l_2^n)/\gamma_p(l_2^n) = \pi_p(l_2^n)/n^{1/2}$.

For the other inequality take T: $l_2^n \to L_p(m)$ as before and observe that $\| Tx \|_2/\| Tx \|_p = \pi_p(l_2^n)/n^{1/2}$.

Remark. Taking $\infty > p \geqslant 2$ and letting $n \to \infty$ in (1) we obtain that the quantity

$$i_{p2}^{**}(l_2) = \lim_{n\to\infty} i_{p2}(l_2{}^n) = 2^{1/2}(\Gamma((p+1)/2)/\pi^{1/2})^{1/p}$$

is equal to $\inf_{\mu,H} \sup_{f\in H} \|f\|_p/\|f\|_2$, the infimum is on all subspaces $H \subset L_p(\mu)$ isometric to l_2 and probability measures μ.

For $1 \leqslant p \leqslant 2$ we get that $2^{-1/2}(\pi^{1/2}/\Gamma((p+1)/2))^{1/p}$ is equal to $\inf_{\mu,H} \sup_{f\in H} \|f\|_2/\|f\|_p$, the infimum is on all subspaces $H \subset L_p(\mu)$ $\gamma_p(l_2)$ complemented and isometric to l_2.

References

1. I. Amemiga and K. Shiga, On tensor products of Banach spaces, *Kotai Math. Sem. Rep.* **9** (1957), 161–178.
2. S. D. Chatterji, Weak convergence in certain special Banach spaces, MRC Technical Sum. Report No. 443, University of Wisconsin, Madison, WI, 1963.
3. J. Cohen, Absolutely p-summing, p-nuclear operators and their conjugates, Dissertation, University of Maryland, 1969.
4. J. Cohen, A characterization of inner product spaces using 2-absolutely summing operators, *Studia Math.* **38** (1970), 271–276.
5. N. Dunford and J. Schwarz, "Linear Operators I," Interscience, New York, 1958.
6. P. Enflo, A counter example to the approximation problem, to appear.
7. T. Figiel, Some remarks on factorization of operators and the approximation problem, to appear.
8. D. J. H. Garling, Absolutely p-summing operators in Hilbert spaces, *Studia Math.* **38** (1970), 319–331.
9. D. J. H. Garling and Y. Gordon, Relations between some constants associated with finite-dimensional Banach spaces, *Israel J. Math.* **9** (1971), 346–361.
10. Y. Gordon, On p-absolutely summing constants of Banach spaces, *Israel J. Math.* **7** (1969), 151–163.
11. Y. Gordon, Asymmetry and projection constants of Banach spaces, *Israel J. Math.* **14** (1973), 50–62.
12. A. Grothendieck, Produits tensoriels topologiques et espaces nucléaires, *Mem. Amer. Math. Soc.* **16** (1955).
13. A. Grothèndieck, Résumé de la theorie métrique des produits tensoriels topologiques, Boletim Soc. Mat. Sao Paulo **8** (1956), 1–79.
14. J. R. Holub, A characterization of subspaces of L_p, *Studia Math.* **42** (1972), 265–270.
15. W. Johnson, Factoring compact operators, *Israel J. Math.* **9** (1971), 337–345.
16. W. Johnson, H. P. Rosenthal, and M. Zippin, On bases, finite dimensional decompositions and weaker structures in Banach spaces, *Israel J. Math.* **9** (1971), 488–506.
17. M. I. Kadec, On linear dimension of the spaces L_p, *Upsehi Mat. Nauk* **13** (1958), 95–98 (Russian).

18. S. Kwapien, On operators factorizable through L_p-spaces, to appear.
19. S. Kwapien, A linear topological characterization of inner product spaces, *Studia Math.* **38** (1970), 277–278.
20. S. Kwapien, On a theorem of L. Schwarz and its applications to absolutely summing operators, *Studia Math.* **38** (1970), 193–201.
21. D. Lewis, Integral operators on $\mathscr{L}_p$-spaces, *Pacific J. Math.*, to appear.
22. J. Lindenstrauss and A. Pelczynski, Absolutely summing operators in $\mathscr{L}_p$-spaces and their applications, *Studia Math.* **9** (1968), 275–326.
23. J. Lindenstrauss and H. P. Rosenthal, The $\mathscr{L}_p$-spaces, *Israel J. Math.* **7** (1969), 325–349.
24. J. S. Morrell and J. R. Retherford, p-trivial Banach spaces, *Studia Math.* **43** (1972), 1–25.
25. L. Nachbin. A theorem of the Hahn-Banach type for linear transformations, *Amer. Math. Soc. Trans.*, **68** (1950), 28–46.
26. H. Nakano, Über normierte teilweisgeordnete Modulen, *Proc. Japan Acad.* **17** (1941), 301–307 (reprinted in the collection of papers "Semi-Ordered Linear Spaces," Tokyo, (1955).
27. A. Persson, On some properties of p-nuclear and p-integral operators, *Studia Math.* **35** (1969), 214–224.
28. A. Persson and A. Pietsch, p-nukleare und p-integrale Abbildungen in Banachräumen, *Studia Math.* **33** (1969), 19–62.
29. A. Pietsch, Absolut-p-summierende Abbildungen in normierten Räumen, *Studia Math.* **28** (1967), 333–353.
30. A. Pietsch, Ideale von S_p-Operatoren in Banachräumen, *Studia Math.* **38** (1970), 59–69.
31. A. Pietsch, Adjungierte normierten Operatorenideale, *Math. Nach.* **48** (1971), 189–211.
32. A. Pietsch, Absolutely p-summing operators in L_p-spaces, I and II, Seminaire Goulaouic-Schwartz 1970–71, June 1971.
33. J. R. Retherford, Operator characterizations of $\mathscr{L}_p$-spaces, *Israel J. Math.*, in print.
34. J. R. Retherford and C. Stegall, Fully nuclear and completely nuclear operators with applications to $\mathscr{L}_1$- and $\mathscr{L}_\infty$-spaces, *Trans. Amer. Math. Soc.* **163** (1972), 457–492.
35. P. Saphar, Produits tensoriels d'espace de Banach et classes d'applications linéaires, *Studia Math.* **38** (1970), 71–100.
36. P. Saphar, Applications p-sommantes et p-decomposantes, *C. R. Acad. Sci.* **270** (1970), 1093–1096.
37. P. Saphar, Hypothèse d'approximation à l'ordre p dans les espaces de Banach et approximation d'applications p absolument sommantes, *Israel J. Math.* **13** (1972), 379–399.
38. R. Schatten, "Norm Ideals of Completely Continuous Operators," Berlin Göttingen Heidelberg, 1960.
39. R. Schatten, "A Theory of Cross Spaces," Princeton Univ. Press, 1950.
40. L. Schwartz, Applications p-radonifiantes et theorème de dualité, *Studia Math.* **38** (1970), 203–213.
41. L. Schwartz, Un théoreme de dualité pour les applications, radonifiantes, *C. R. Acad. Sci.* **268** (1969), 1410–1413.
42. A. Tong, Diagonal nuclear operators on l_p-spaces, *Trans. Amer. Math. Soc.* **143** (1969), 235–247.

43. A. Zygmund, "Trigonometric Series," Vol. 1, Cambridge Univ. Press, London New York, 1959.
44. H. P. Rosenthal, On the subspaces of $L_p (p > 2)$ spanned by sequences of independent random variables, *Israel J. Math.* **8** (1970), 273–303.
45. J. Lindenstrauss, On the extension of operators with a finite dimensional range, *Illinois J. Math.* **8** (1964), 488–499.
46. M. I. Kadec and A. Pelczynski, Bases, lacunary sequences and complemented subspaces in the spaces L_p, *Studia Math.* **21** (1962), 161–176.

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