(Knowledge for Development)

KIBABII UNIVERSITY

UNIVERSITY EXAMINATIONS

2020/2021 ACADEMIC YEAR

THIRD YEAR FIRST SEMESTER

MAIN EXAMINATION

FOR THE DEGREE OF BACHELOR OF SCIENCE

COURSE CODE: MAP 312

COURSE TITLE: **LINEAR ALGEBRA III**

DATE: 26/05/2022 **TIME:** 9:00 AM - 11:00 AM

INSTRUCTIONS TO CANDIDATES

Answer Question One and Any other TWO Questions

TIME: 2 Hours

QUESTION ONE (30marks)

(a). Define the following terms

(i). Complex vector in $\mathbb{C}^n$ (1 mk)

(ii). Nilpotent matrix (1 mk)

(iii). Orthogonal matrix (1 mk)

(b). (i). Let $\mathbb{C}^n$ be a complex vector space $u, v \in \mathbb{C}^n$ and $k \in K$, K a scalar field. If $\bar{u}$, and $\bar{v}$ denotes the conjugates of u and v respectively, show that $\overline{ku} = \bar{k}\bar{u}$ and $\overline{u - v} = \bar{u} - \bar{v}$.(6mks)

(ii). Let $u, v \in \mathbb{C}^n$ with $u = (4 - i, i - 2, -i)$ and $v = (4 + i, -2i, 3)$. Find the Euclidian norm $||u||$ of u and the Euclidean inner product $\langle u, v \rangle$. (4 mks)

(c). (i). Given that x is the eigenvector of a nonzero matrix A corresponding to an eigenvector λ. Explain why $x \neq 0$. (2 mks)

(ii). Determine the eigenvalues and eigenvectors for the following matrix

$$\begin{bmatrix} 4 & -2 \\ 10 & -4 \end{bmatrix}$$ (5 mks)

(d). Prove that an orthogonal matrix is Isometric (4 mks)

(e). Make a change of variable to transform the quadratic form $Q(x_1, x_2) = 2x_1^2 - 6x_1x_2 + 2x_2^2$ into a quadratic form with no cross-product terms. (6 mks)

QUESTION TWO (20 MKS)

(a) Show that an inner product defined by $f(u, v) = \sum_{i=1}^{n} u_i \, \overline{v_i}$ is a complex valued bilinear form. (4 mks)

(b). Determine all possible Jordan Canonical forms J for a linear operator $T: V \to V$ whose characteristic polynomial $\Delta(t) = (t - 3)^5$ and whose minimum polynomial $m(t) = (t - 3)^2$. (4 mks)

(c). Let A and B be linear operators on complex vector space V such that $A: V \to V$ and $A: V \to V$. If k a complex number prove that

i). $(A^*)^* = A$ (2 mks)
ii). $(A + B)^* = A^* + B^*$ (2 mks)
iii). $(AB)^* = B^*A^*$ (2 mks)

(d). Prove that the eigenvalues of real symmetric matrices are real and the eigenvectors are orthogonal. (6 mks)

QUESTION THREE (20 MKS)

(a). (i). What is a bilinear form? (2 mks)

(ii). Classify the quadratic form $g(x_1, x_2) = 4x^2 - 2x_1x_2 + 4x_2^2$ as either positive definite, negative definite or indefinite. (3 mks)

(b). Show that the eigenvectors from different Eigen spaces of a Hermitian matrix are orthogonal. (5 mks)

(c). Prove that $\langle x, y\rangle = \frac{1}{4}\{\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\}$ where $i = \sqrt{-1}$. (5 mks)

(d). Let $\langle Ax \cdot Ay\rangle = \langle x \cdot y\rangle$ for all $x, y \in \mathbb{R}^n$. Show that A is orthogonal. (5 mks)

QUESTION FOUR (20 MKS)

(a). If λ is an eigenvalue of real $n \times n$ matrix A, and if x is the corresponding eigenvector, then $\bar{\lambda}$ is also an eigenvalue of A and $\bar{x}$ is a corresponding eigenvector. (3 mks)

(b). If A is unitary then, show that

(i). A is an isometry. (4 mks)

(ii). Under what conditions is the matrix $\begin{bmatrix} x & 0 & 0 \\ 0 & 0 & z \\ 0 & y & 0 \end{bmatrix}$ unitary? (3 mks)

(c). Prove that if T is an $n \times n$ orthogonal matrix, then the row and well as column vectors of T forms an orthonormal set in $\mathbb{R}^n$ with the Euclidean inner product (5 mks)

(d). Prove that $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} |\lambda| & 0 \\ 0 & |\lambda| \end{bmatrix}\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$ where $\lambda = a \pm bi, i = \sqrt{-1}$ are the eigenvalues and α is an angle from positive x-axis to the ray that joins the origin to the point (a, b) where a and b are not both zeros. (5 mks)

QUESTION FIVE (20 MKS)

(a). Define a symmetric matrix giving an example (2 mks)

(b). Prove that unitarily diagonalizable matrix must be Hermitian. (3 mks)

(c). Orthogonally diagonalizable matrix A, $A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$ (15 mks)