(*Knowledge for Development*)

KIBABII UNIVERSITY

UNIVERSITY EXAMINATIONS

2021/2022 ACADEMIC YEAR

FIRST YEAR FIRST SEMESTER

MAIN EXAMINATION

FOR THE DEGREE OF MASTER OF SCIENCE IN STATISTICS

COURSE CODE: **STA 806**

COURSE TITLE: THEORY OF **LINEAR MODELS**

DATE: 26/05/2021 **TIME:** 2:00 PM - 4:00 PM

INSTRUCTIONS TO CANDIDATES

Answer Question One any other Two Questions

TIME: 2 Hours

QUESTION ONE (30 MARKS)

(a). Consider a linear regression model.

Show that the model can be written in matrix form as

$\underline{Y} = X\beta + \underline{\varepsilon}$

Where $\underline{Y}$, β and $\underline{\varepsilon}$ are vectors of order n×1; (k+1)×1 and n×1 respectively, while X is a matrix of order n×(k+1). (4marks)

$\underline{\hat{\beta}} = (X^T X)^{-1} X^T Y$ (4marks)

(b) Let $S^2 = \frac{1}{n-k-1}\sum_{i-1}^{n}(y_i - \underline{X}_i^T \underline{\hat{\beta}})^2$ where $\underline{X}_i^T$ is the i-th row of the matrix X. Show that if

$\text{Var}(\underline{\varepsilon}) = \sigma^2 I$ then $E(S^2) = \sigma^2$ (6marks)

(c). In part (b) Let $\hat{Y} = \hat{\beta}_0 + \hat{\beta}_1 X$ be the predicted value of Y. Let $\underline{X}^T = (1 \;\; X)$ such that $\hat{y} = \hat{\beta}_x^T \underline{X} + \varepsilon$

Show that if $\underline{Z}^T = (1 \;\; cx)$ where c $\hat{y} = \hat{\beta}_x^T \underline{X}$ where $\hat{\beta}_z$ is the Least square estimator of $\underline{\beta}$ assuming $Y = \underline{\beta}^T \underline{Z} + \varepsilon$ (6 marks)

(d). Let $\underline{Y} = \beta_0 + \beta_1 X + \varepsilon$ where ε is the error term. Using results in part (b) or, otherwise deuce the least square estimators of β_o and β_1 say $\hat{\beta}_0$ and $\hat{\beta}_1$ respectively. Show that

(i). $E(\hat{\beta}_0) = \beta_o$

(ii). $E(\hat{\beta}_1) = \beta_1$

Determine

(iii). Var $(\hat{\beta}_0)$

(iv). Var $(\hat{\beta}_1)$

(v). Cor $(\hat{\beta}_0, \hat{\beta}_1)$ (10 marks)

QUESTION TWO(20 MARKS)

(a). Show that if E(Y) =X $\underline{\beta}$ and Cor ($\underline{Y}$) = $\sigma^2 I$ then the least square estimators $\hat{\beta}_j$,

j = 0,1,…,k, have minimum variance among all linear unbiased estimators. (10 marks)

(b). Using part (a) or otherwise, show that if E($\underline{Y}$) = X$\underline{\beta}$ and Cov($\underline{Y}$) = $\sigma^2 I$, then the best linear unbiased estimator of $\underline{a}^T\underline{\beta}$ is $\underline{a}^T\hat{\underline{\beta}}$ where $\hat{\underline{\beta}}$ is the least square estimator of $\underline{\beta}$. (5marks)

(c). Does the results in part (a) rely on the distribution of the random vector $\underline{Y}$? Comment

(5marks)

QUESTION THREE (20 MARKS)

(a). Show that if $\underline{Y}$ ~ N_n[X $\underline{\beta}$, $\sigma^2 I$] where X is n×(k+1) matrix of rank k+1<n, the maximum likelihood estimator of $\underline{\beta}$ and σ^2 are $\hat{\underline{\beta}}$ = $(X^TX)^{-1}X^T\underline{Y}$

and $\hat{\sigma}^2 = \frac{1}{n}(\underline{Y} - X\hat{\underline{\beta}})^T(Y - X\hat{\underline{\beta}})$ (10 marks)

(b). Using the results in part (a), or otherwise, Show that

(i). $\hat{\underline{\beta}}$ is N_{k+1} [$\underline{\beta}$, $\sigma^2 (X^TX)^{-1}$]

(ii) $n\frac{\hat{\sigma}^2}{\sigma^2}$ is chi-square with degrees of freedom = n-k-1

(iii). $\hat{\underline{\beta}}$ and $\hat{\sigma}^2$ are independent . (10 marks)

QUESTION FOUR(20 MARKS)

Consider the data in the following table

Observation Number	Y	X_1	X_2
1	2	0	2
2	3	2	6
3	2	2	7
4	7	2	5
5	6	4	9
6	8	4	8
7	10	4	7
8	7	6	10
9	8	6	11
10	12	6	9
11	11	8	15
12	14	8	13

(a). Show how the date can be modeled by a regression model given by

$\underline{Y} = X\,\underline{\beta} + \underline{\varepsilon}$ (5marks)

(b). Compute the least square estimate of $\hat{\underline{\beta}}$. (5marks)

(c). If $\text{Var}(\underline{\varepsilon}) = \sigma^2 I$ is known, calculate $\text{Var}(\hat{\underline{\beta}})$. (5marks)

(d). Calculate the estimate of the estimator S^2, defined in question one (b).

(5 marks)

QUESTION FIVE (20 MARKS)

For the multiple regression model given by $\beta' = [\beta \;\; \beta_1 \;\; \beta_2]$ and

$$X = \begin{matrix} 1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{matrix}$$

(a) Find rank(X);

(b) Find a generalized inverse of X X;

(c) Hence find a least squares estimator of β;

(d) Check whether or not β_1 is estimable;

(e) check whether or not $\beta_1 + \beta_2$ is estimable.