(Knowledge for Development)

KIBABII UNIVERSITY

UNIVERSITY EXAMINATIONS

2021/2022 ACADEMIC YEAR

FORTH YEAR SECOND SEMESTER

MAIN EXAMINATION

FOR THE DEGREE OF BACHELOR OF SCIENCE

COURSE CODE: **MAP 422.**

COURSE TITLE: **DIFFERENTIAL TOPOLOGY**

DATE: 30/08/2022 **TIME:** 9:00 AM - 11:00 AM

Answer question ONE and any other TWO Questions

TIME: 2 Hours

QUESTION ONE (30 MARKS)

a). Define the following terms

(i). A manifold (2 mks)

(ii). Homeomorphism (2 mks)

(iii). transversally intersecting manifolds (2 mks)

(iv). Tangent space (2 mks)

b). A cone is not a manifold. Explain. (3 mks)

c). Let $f: X \to Y$ be a smooth function. Define the critical values of f, hence use the Sards theorem, and determine its measure (3 mks)

d). Prove that $f(t) = \sin 2t$ is a smooth map. (3 mks)

e). Prove that a subset of $\mathbb{R}^n$ is a manifold (4 mks)

f). For which values of a does the hyperboloid $x^2 + y^2 - z^2 = 1$ intersect the sphere $x^2 + y^2 + z^2 = a$ transversally? What does the intersection look like for different values of a? (5 mks)

g). Given that $f: X \to Y$ is a smooth map with regular value $y \in Y$, prove that $f^{-1}(y)$ is a submanifold of X. (4 mks)

QUESTION TWO (20 MARKS)

a). Differentiate between immersion and submersion (4 mks)

b). State the inverse function theorem hence, use it to show that the map $f(t) = (r\cos t, r\sin t)$ for $r > 0$ is a local diffeomorphism but not a global diffeomorphism. (6 mks)

c). If the smooth map $f: X \to Y$ is transversal to a submanifold$Z \subset Y$, prove that the pre-image $f^{-1}(Z)$ is a submanifold of X. (6 mks)

d). Let Z be a pre-image of a regular value $y \in Y$ under the smooth map $f: X \to Y$. Then the kernel of the derivative $df_x: T_x(X) \to T_y(Y)$ at any point $x \in Z$ is precisely the tangent space Z, $T_x(Z)$. (4 mks)

QUESTION THREE (20 MARKS)

a). Define the differential df_x hence express it in terms of the derivative of two parametrizations to its domain and codomain. (3 mks)

b). Let $S^1 \subset \mathbb{C}$ be the set $\{z \in \mathbb{C} : |z| = 1\}$. Define a map $F: S^1 \to S^1$ by $z \mapsto z^2$ where $z = x + iy$ for $i = \sqrt{-1}$. Determine the differential df at i. (11 mks)

c). Given that $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ are smooth maps of manifolds. Prove that $d(g \circ f)_x = dg_{f(x)} \circ df_x$. Use commutive diagrams for Illustration. (6 mks)

QUESTION FOUR (20 MARKS)

a). Define four parameterizations that cover the unit circle $S' = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. (6 mks)

b). Define the term an embedding hence prove that an embedding $f: X \to Y$ maps X diffeomorphically into a submanifold of Y. (8 mks)

c). Let X and Y represent the x- and y-axes of the xy-plane. Since X and Y are manifolds, prove that the $X \times Y$ is also a manifold and its dimension is $\dim X + \dim Y$. (6 mks)

QUESTION FIVE (20 MARKS)

a). Define the term chart hence find all charts that cover the manifold $\mathbb{R}^2$(the Euclidian manifold of dimension 2) (4 mks)

b). Prove that the dimension of the tangent space $T_x(X)$ is equal to that of the manifold X. (5 mks)

c). Determine the tangent space to the paraboloid defined by $x^2 + y^2 - z^2 = a$ at $(\sqrt{a}, 0,0)$. (8 mks)

d). Prove that $f(x) = x^2$ is a smooth function but not a global diffeomorphism. (3 mks)