(3)

(Knowledge for Development)

KIBABII UNIVERSITY

UNIVERSITY EXAMINATIONS

2021/2022 ACADEMIC YEAR

FOURTH YEAR SECOND SEMESTER

MAIN EXAMINATION

FOR THE DEGREE OF BACHELOR OF EDUCATION (SBD)

COURSE CODE: MAT 406

COURSE TITLE: FIELD THEORY

DATE: 06/09/2022 **TIME:** 2:00 PM - 4:00 PM

INSTRUCTIONS TO CANDIDATES

Answer Question One and Any other TWO Questions

TIME: 2 Hours

QUESTION ONE COMPULSORY (30 MARKS)

a) Define the following terms (5marks)
   i. Ring (2marks)
   ii. Field extension (2marks)
   iii. Field

b) Let $f(x) = 3x^2 + 2$, $g(x) = 4x^4 + 2x^3 + 6x^2 + 4x + 5$ in $\mathbb{Z}_7[x]$, find $q(x)$ amd $r(x)$ such that $g(x) = f(x)q(x) + r(x)$ (6marks)

c) Describe the splitting fields of the polynomial $x^2 - 2$ in $\mathbb{Q}[x]$ (2marks)

d) Prove that given a field $\mathbb{F}$, $\mathbb{F}(\alpha, \beta) = (\mathbb{F}(\alpha))(\beta)$ (5marks)

e) Let $p(x) = 3x^2 + x + 2$, $q(x) = 2x^3 + x^2 + 5$ in $\mathbb{Z}_6[x]$. Find their product and hence the $deg(p(x).q(x))$ (8marks)

QUESTION TWO (20 MARKS)

a) Define the following terms (2marks)
   i. Kernel (2marks)
   ii. Characteristic (2marks)
   iii. Minimal polynomial (2marks)
   iv. Associate polynomial

b) Find the gcd of $f(z) = z^4 + 4z^3 + 5z^2 + 4$ and $g(z) = z^2 + 5z + 4$ in $\mathbb{Z}_7(x)$ (8marks)

c) Prove that if R is an integral domain and p(x) and q(x) are non-zero elements of R[x], then $deg(p(x).q(x)) = degp(x) + degq(x)$ (4marks)

QUESTION THREE (20 MARKS)

a) Describe the splitting fields of the following polynomial (3marks)
   i. $f(x) = x^4 + x^2 - 2$ in $\mathbb{Q}[x]$

b) Discuss the minimal polynomials of the following (3marks)
   i. $\sqrt{2}$ over $\mathbb{Q}$ (3marks)
   ii. $\sqrt[3]{4}$ over $\mathbb{Q}$

c) What is the splitting field for $f(x) = x^4 + 4$ over $\mathbb{Q}$ (5marks)

d) Prove that for any commutative ring R with unity, the ring $R[x]$ of polynomials over R contains a subring R' that is isomorphic to R (6marks)

QUESTION FOUR (20 MARKS)

a) Define the following terms

   i. Degree of a field extension (2marks)

   ii. Homomorphism (2marks)

   iii. Unit (2marks)

b) Determine whether $f(x) = 2x^3 + x^2 - 5x + 2$ is irreducible over $\mathbb{Z}_6$ (4marks)

c) Using the rational root test, express $f(x) = 3x^3 - 4x^2 - 17x + 6$ as a product of irreducible polynomials. (10marks)

QUESTION FIVE (20 MARKS)

a) Prove that if $f(x)$ is a polynomial of degree 2 or 3 over a field F then $f(x)$ is irreducible over F iff $f(x)$ has no zeros in F (6marks)

b) Determine whether $f(x) = 2x^6 - 3x^4 + 6x^2 - 6x + 12$ is irreducible or not using the Einstein criterion (4marks)

c) Prove that the characteristic of a field $\mathbb{F}$ is either zero or a prime p (5marks)

d) Find the zero of the polynomial $f(x) = x^2 + 1$ in $\mathbb{Z}_5$ (5marks)