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Article

Distribution of Spectrum in a Direct Sum Decomposition of Operators into Normal and Completely Non-Normal Parts

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Abstract: We discuss the distribution of spectra of a direct sum decomposition of an arbitrary operator into normal and completely non normal parts. We utilize the fact that any given operator $T \in B(H)$ can be decomposed into a direct summand $T = T_1 \oplus T_2$ with T_1 and T_2 are the normal and completely non normal parts respectively. This canonical decomposition is preferred to other forms of decomposition such as Polar and Cartesian decompositions because these two do not transfer certain properties (for instance the spectra, numerical range, and numerical radius) from the original /decomposed operator to the constituent parts. This is presumably done since these parts are simpler to deal with.

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1. Preliminaries

1.1. Notation and Terminology

In this paper, a Hilbert space will be denoted by a capital letter H, while a bounded linear operator shall be denoted by T, where an operator means a bounded linear transformation (equivalently, a continuous linear transformation) $T: H \to K$. $B(H)$ denotes the set of bounded linear transformations from H into itself, which is equipped with the (induced uniform) norm. For an operator T, we denote by $T^*$the adjoint of T. The *spectrum* of T is defined and denoted by $\sigma(T) = \{\lambda \in \mathbb{C}: \lambda I - T$ not invertible$\}$. It is a union of disjoint components, namely, the *point spectrum* $\sigma_p(T) = \{\lambda \in \mathbb{C}: \lambda I - T$ is not injective$\}$, the *continuous spectrum* $\sigma_c(T) = \{\lambda \in \mathbb{C}: \lambda I - T$ is injective and $\lambda I - T$ has a dense range$\}$ and the *residual spectrum* $\sigma_R(T) = \{\lambda \in \mathbb{C}: \lambda I - T$ is injective and $\lambda I - T$ has a non-dense range$\}$.
$\sigma_{ap}(T)$ shall denote the *approximate point spectrum* defined by $\sigma_{ap}(T) = \{\lambda \in \mathbb{C}: \lambda I - T$ not bounded$\}$.

An operator T is said to be:

An *isometry* if $T^*T = I$, *Unitary* if $T^*T = TT^* = I$, *Hyponormal* if $T^*T \geq TT^*$,
p-hyponormal if $(T^*T)^p \geq (TT*)^p$ where $0 < p < 1$, *(p,k)-quasihyponormal* if
$T^*[(T^*T)^p - (TT^*)^p]T^k \geq 0$ for some positive integer k and $0 < p \leq 1$ and
a *unilateral shift* if there exist a sequence $\{H_0, H_1, \dots\dots\dots\}$ of pairwise orthogonal subspaces of H such that:

- $H_0 \oplus H_1 \oplus \dots\dots$
- T spans H_n isometrically onto H_{n+1}.

For a subspace M of H, the orthogonal complement of M is given by $M^\perp = \{u \in H: < u, v >= 0$ for all $v \in M\}$.

2. Introduction

We first give some results concerning the spectrum of a normal operator.

Definition 2.1 An operator $T \in B(H)$ is said to be *normal* if $T^*T = TT^*$ (equivalently, if $\| Tx \| = \| T^*x \| \; \forall x \in H$).

Definition 2.2 Let H be a Hilbert space, a subspace M of H is said to be *invariant* under an operator $T \in B(H)$ if $TM \subset M$ or precisely $Tx \in M$ *for all* $x \in M$.

We can then state the following result:

Corollary 2.3

An operator $T \in B(H)$ is *invariant* under M iff T^* is invariant under $M^\perp$.

Definition 2.4

A subspace $M \subset H$ is said *to reduce* an operator $T \in B(H)$ if M is invariant under both $T$and T^*.

We state and prove the following lemma.

Lemma 2.5

A subspace $M \subset H$ is said to reduce an operator if both M and $M^{\perp}$ are invariant under T.

Proof

Suppose M reduces T, then M is invariant under both T and T^*. In particular, M is invariant under T^*implying by the corollary above that $M^{\perp}$ is invariant under $T^{**} = T$. Thus, both $M$and $M^{\perp}$are invariant under T.

Conversely, suppose both M and $M^{\perp}$ are invariant under T, then by the corollary 2.3 above, $M^{\perp}$ invariant under T imply that M is invariant under T^*. Therefore M and $M^{\perp}$ are invariant under T and by definition 2.4, M reduces T.

Lemma 2.6

If T is a normal operator, then $\sigma_R(T) = \emptyset$.

Proof

Suppose $\sigma_R(T) \neq \emptyset$ and let $\lambda \in \sigma_R(T)$.

By definition, $\lambda \in \sigma_R(T)$ if $(\lambda I - T)^{-1}$ exist as a map bounded or unbounded but actually means that there exist a non zero vector x such that

$(\bar{\lambda} I - T^*)x = 0$…………………………………………………………………………..(1)

Since T is normal, so is $\lambda I - T$. Equivalently,

$\|(\lambda I - T)x\| = \|(\bar{\lambda} I - T^*)x\|$ for all $x \in H$……………………………………………..…(2)

From these two arguments, (1) and (2);

$\|(\lambda I - T)x\| = 0$ for $x \neq 0$ or $(\lambda I - T)x = 0$ for $x \neq 0$

Therefore $x \in \sigma_p(T)$.

This is a contradiction since $\sigma_R(T) \cap \sigma_p(T) = \emptyset$. Therefore, $\sigma_R(T) = \emptyset$.

Corollary 2.7

If T is a normal operator, then $\sigma_{ap}(T) = \sigma(T)$.

Proof

From Lemma 2.6 together with the definitions of $\sigma_{ap}(T)$ it implies that;

$\sigma_{ap}(T) \supseteq \sigma_p(T) \cup \sigma_c(T)$ and since $\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_R(T)$

Then the result follows by the above corollary.

3. On a Direct Sum of a Quasinormal Operator

ZbigniewBurdak (2013), classifies an operator $T \in B(H)$ as a quasinormal if T commutes with T^*T i.e. $T(T^*T) = (T^*T)T$. Quasinormal operators were first studied by Brown (1953) and it's quite clear that quasinormal $\supset$ normal and thus T can be quasinormal but not normal as illustrated below;

Example 3.1

Let $H = l_2$ and T be the unilateral shift given by the following matrix

$$T = \begin{pmatrix} 0 & & \\ & \cdots & 0 \\ 1 & & \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \\ & & \end{pmatrix}$$

Then

$$T^*T = I \Rightarrow T(T^*T) = T = (T^*T)T$$

Hence T is quasinormal. However $T^*T - TT^* = diag\ (1,0,0, \ldots \ldots . .)$ hence T is not normal.

Notice that if T is quasinormal so are the powers $T^n$for $n = 0,1,2, \ldots$....Brown (1953), showed that every quasinormal operator can be written as a direct sum $T = N \oplus S$ where N is the normal part and S the dilated shift operator associated with a positive operator. On the study of their spectra, the spectrum of T has an interior part. S is actually a tensor product and thus $\sigma(S)$ is a closed disk given by$\{z: |z| \leq \|S\|\}$.

If $c > 0$, then cT where T, the unilateral shift illustrated in the example above is completely quasinormal with spectrum $\{z: |z| \leq c$. Thus in conclusion, a compact set X is the spectrum of a completely quasinormal operator T if and only if $X = \{z: |z| \leq c\}$; for $c > 0$.

Lemma 3.2

If T is a (p, k)-quasihyponormal operator, then T has the following matrix representation; $T = \begin{bmatrix} T_1 & T_2 \\ 0 & T_3 \end{bmatrix}$ where T_1 is a p-hyponormal operator on $\overline{Ran(T^k)}$ and $T_3^k = 0$.

Furthermore, $\sigma(T) = \sigma(T_1) \cup \{0\}$.

4. On a Direct Summand of a Quasi-*Paranormal Operator

Definition 4.1 Arora and Thukral (1986)

An operator $T \in B(H)$ is said to be *quasi-*paranormal* if for each vector$x \in H$, $\|T^*T\|^2 \leq \|T^2x\|\|x\|$.

We then state the following results;

Definition 4.2

Let $T \in B(H)$be a quasi-*paranormal operator, then for any scalar λϵℂ: $N_T(\lambda) = N_T(\bar{\lambda})$

If we let $N = N_T(\lambda)$, then N reduces T and T restricted to N is normal. Furthermore $N_T(\lambda) \perp N_T(\mu)$ whenever $\lambda \neq \mu$.

Theorem 4.3

Let $T \in B(H)$ be a quasi-* paranormal operator, then T can be expressed uniquely as a direct sum $T = T_1 \oplus T_2$ defined on the product space $H = H_1 \oplus H_2$ such that the following properties are satisfied;

- T_1 is normal
- T_2 is a quasi-*paranormal and $\sigma_p(T_2) = \emptyset$.

Proof

Let $H_1 = \oplus_{\lambda \in \sigma_{p(T)}} N_T(\lambda)$, then H_1 is spanned by proper vectors of T. Since $N_T(\lambda)$, is a closed subspace, H_1 is thus a closed linear subspace and therefore $H = H_1 \oplus H_1^{\perp} = H_1 \oplus H_2$ where $H_2 = H_1^{\perp}$.

Let T_1 be T restricted to H_1 and T_2 be Trestricted to H_2 .

Therefore we can write $T = T_1 \oplus T_2$ uniquely.

Let $x \in H_1$ then, $x = x_\lambda + x_\mu + \cdots$ where $x_\lambda \in N_T(\lambda)$ and $x_\mu \in N_T(\mu)$ etc.

Therefore, $T_1^* T_1(x) = T_1^* T_1(x_\lambda + x_\mu + \cdots) = \lambda T_1^* x_\lambda + \mu T_1^* x_\mu + \cdots = \lambda\bar{\lambda} x_\lambda + \mu\bar{\mu} x_\mu + \cdots = \bar{\lambda}\lambda x_\lambda + \bar{\mu}\mu x_\mu + \cdots = \bar{\lambda} T_1 x_\lambda + \bar{\mu} T_1 x_\mu + \ldots = T_1 T_1^*(x_\lambda + x_\mu + \cdots) = T_1 T_1^*(x)$

Hence T_1 is normal.

Let $x \in H_2$, then $x = 0 + x \in H_1 \oplus H_2$. Since T is quasi-*paranormal, then $\|T_2^* T_2(x)\| = \|T^* T(0 + x\| \leq \|T^3(0 + x\| \|T(0 + x\| = \|T_2^3(x)\| \|T_2(x)\|$ for all $x \in H$

Now suppose $\sigma_p(T_2) \neq \emptyset$, if $\mu \in \sigma_p(T_2)$, then there is a non-zero vector x in H_2 such that $T_2 x = \mu x$.

Let $T(0 + x) = T_2 x = \mu x = \mu(0 + x)$, then $x = 0 + x \in N_T(\mu)$ implying that $x \in H_1$. This is a contradiction since x is non zero. Therefore $\sigma_p(T_2) = \emptyset$.

Proposition 4.4 (Wold decomposition), Faulkner and Huneycutt (1978)

Every isometry is a direct sum of aunitary operator and a unilateral shift.

Proposition 4.5

An isometry is completely non normal (c.n.n.) or pure if and only if it is a unilateral shift.

Proof

This is trivially true from the inclusion $unitary \subset normal$

The following example clearly illustrates the above proposition.

Example 4.6

Let $H = l_2$, the space of all square-summable sequences and A the right shift operator, $A(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$. Then $\|Ax\| = \|x\|$ for all $x \in l_2$.

Since A^* is the left shift operator, we have that $A^*(A(x)) = A^*(0, x_1, x_2, \ldots) = (x_1, x_2, \ldots) = x$

On the other hand, we have $A\left(A^{*}(x)\right) = A(x_2, x_3, \dots) = (x_2, x_3, \dots) \neq x$

Thus $A^{*}A \neq AA^{*}$ implying that A is not normal. Therefore a right shift operator is hyponormal but not normal (has no direct summand).

Corollary 4.7

If A is a hyponormal operator and $\lambda \in \ \sigma_p(A), then\ Ker\ (A-\lambda)$ reduces A.

Corollary 4.8

If A is a pure hyponornormal operator, then $\sigma_p(A) = \emptyset$.

Lemma 4.9

Let A be a p-hyponormal operator for $0 < p < \frac{1}{2}$, then A has a normal summand if and only if $\tilde{\tilde{A}}$ has a normal summand.

Thus it can be shown that if A is a p-hyponormal operator, then A is normal iff $\tilde{\tilde{A}}$ is normal and the point spectrum of A consists of normal eigenvalues.

Lemma 4.5

If A is a pure p-hyponornormal operator, then $\sigma_p(A) = \emptyset$.

Proof

Suppose that $\sigma_p(A) \neq \emptyset$, then since $0 \in \sigma_p(A)$, it implies that 0 is a normal eigenvalue of A.

We may assume $0 \notin \sigma_p(A)$.

Let$\lambda \in \sigma_p(A), \neq 0$ and let x be an eigenvector corresponding to the eigenvalue λ, then,

$(A-\lambda)x = 0$ imply that $\left\{|A|^{\frac{1}{2}}U|A|^{\frac{1}{2}} - \lambda\right\}|A|^{\frac{1}{2}}x = 0$

$\Rightarrow \langle A \triangleq \lambda\rangle |A|^{\frac{1}{2}}x = 0$ implying $\langle |\hat{A}|^{\frac{1}{2}}V|\hat{A}|^{\frac{1}{2}} - \lambda\rangle\ |\hat{A}|^{\frac{1}{2}}x = 0$ and thus $\langle A \triangleq \lambda\rangle |\hat{A}|^{\frac{1}{2}}|\hat{A}|^{\frac{1}{2}}x = 0$

That is $\lambda \in \sigma_p\left(\tilde{\tilde{A}}\right) = \emptyset$. Since $\tilde{\tilde{A}}$ is hyponormal, is a normal eigenvalue of $\tilde{\tilde{A}}$, Daoxing (1981).

By lemma 4.5 above, it implies that A has a normal direct summand hence a contradiction since A is pure. Therefore, $\sigma_p(A) = \emptyset$.

5. Conclusion

Based on the basic notations and definitions in sections 1 and 2, as one of our main results concerning the spectrum of a normal operator, in Lemma 2.6, we showed that a bounded linear operator T is normal if $\sigma_R(T) = \emptyset$. This result was further extended in Corollary 2.7, where it was proved that if T is a normal operator, then $\sigma_{ap}(T) = \sigma(T)$.

In section 3, by classifying an operator $T \in B(H)$ as a quasinormal, evidently, it was shown that quasinormal $\supset$ normal and in Example 3.1, we illustrated that the spectrum of T can be decomposed as a direct sum if T is quasinormal but not normal. Here, we concluded that a compact set X is the spectrum of a completely quasinormal operator T if and only if $X = \{z: |z| \leq c\}$; for $c > 0$.

Following the definition of an operator $T \in B(H)$ being *quasi-*paranormal* by Arora and Thukral (1986), in Theorem 4.3, we proved that T can be uniquely expressed as a direct sum $T = T_1 \oplus T_2$ such that T_1 is normal and T_2 is a quasi-*paranormal with $\sigma_p(T_2) = \emptyset$. In Proposition 4.5, using Example 4.6, we showed that an isometry is completely non normal (c.n.n.) or pure if and only if it is a unilateral shift. Consequently, in Lemma 4.5, it followed that for a right shift operator A, if A is a pure p-hyponornormal operator, then $\sigma_p(A) = \emptyset$.

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