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Stability Bifurcation Analysis of a Fishery Model with Nonlinear Variation in Market Price

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Abstract

We develop a fishery model with price dependent harvesting by formulating a system of three differential equations describing its dynamics. Assuming that the price of the harvested fish on the market evolves relatively faster than the evolution of the fish stock and the fishing effort, we apply approximate aggregation to reduce the system of equations from three to two. From the stability analysis of the aggregated model, we show the co-existence of three strictly positive equilibria where two are stable and are separated by a saddle. The two stable equilibtia represent two kinds of fishery namey; an over-exploited fishery where the fishery supports a large economic activity but risks extinction and an under-exploited fishery where the stock is maintained at a large level far from extinction but the fishery only supports a small economic activity.

Mathematics Subject Classification: 93A30, 92B05, 34C23

Keywords: Aggregation, Equilibrium points, Local stability

1 Introduction

Sustainable harvesting of renewable resources has been widely considered in bioeconomics in an attempt to have sustainable resources exploitation, see for instance [3], [4] and [12]. A fishery is one of these resource and fishery dynamics has been studied with earlier models considering population growth described by the Verhulst's growth model without harvesting. When we take harvesting into consideration, the factors that affect the harvesting effort become crucial in any meaningfull model. The basic harvesting effort drive in a fishery is the market price of the fish that is necessarily a function of supply and demand. Thus the most feasible time continous model describing the relationship between the three main variables and parameters to study the fishery dynamics with price dependent harvesting is of the form

$$\begin{aligned} \dot{n} &= f(n) - h(n,E), \\ \dot{E} &= \beta(ph(n,E) - cE), \\ \dot{p} &= \alpha p(D(p) - h(n,E)), \end{aligned} \tag{1}$$

where $n := n(t)$ and $E := E(t)$ represents the population of fish and fishing effort respectively at time t. The variable $p := p(t)$ is the price of the stock of fish at any time t. The constant β is a positive adjustment coefficient depending on the fishery and the landed fish price p per unit of the landed stock at time t. The constant α is referred to as the price adjustment parameter.

The first equation of Equation(1) describes the rate of growth of the fish resource which is harvested. The function

$$f(n) = rn(1 - \frac{n}{k}), \tag{2}$$

is the logistic growth rate function, where r is the intrinsic growth rate of the fish while k is the carrying capacity and reflects the level to which the fish will grow if there is no harvesting. The function $h(n, E)$ is the harvesting function that depends on the fish resource and the fishing effort and hence mimicks predation, see for instance [10]. Thus $h(n, E) = g(n, E)E$, where the function $g(n, E)$ is the amount of fish captured per unit of fishing effort. A suitable function, commonly used in fishery management models, tha takes into account mortality and harvesting effort as control variables to stock growth is the schaefer function, see for instance Arne Eide *et al,* [1]. We thus choose $g(n, E) = qn$, where q is a positive constant referred to as capturability. Hence

$$h(n, E) = qnE. \tag{3}$$

The second equation of Equation (1) describes the evolution of the fishing effort that depends on the difference between the benefit and the cost of fishing effort. The plausible relation describing the dynamics of harvesting is

$$\dot{E} \propto \text{(benefit - cost)}.$$

The total benefit is the product of the market price p and the total catch $h(n, E)$, while the total cost is the product of the cost per unit of fishing c and the fishing effort E. Thus with $h(n, E)$ given in Equation(3), we have

$$\dot{E} = \beta E(pqn - c). \tag{4}$$

The third equation in Equation (1) describes the variation of market price p, that depends on the demand, the supply of the fish and, the price dynamics. We assume that relative variations in the market price are governed by a simple balance between demand and supply of fish. This relation can be represented by

$$\dot{p} = \alpha p(D(p) - S(p)),$$

where $D(p)$ and $S(p)$ denote the demand and supply functions of the fish respectively, see for instance [6]. The arqument of the demand schedule is taken as $p(t)$ in keeping with the simplest assumption that consumers base all buying decisions on the current market price. A linear demand function dependent on the market price p is

$$D(p) = A - p(t),$$

where A is a positive constant parameter representing the limit threshold of the market price, see for instance [5], is chosen as a demand function, such that the demand decreases linearly with increasing price. This suits price sensitive resources in which the marginal cost is unit and their consumption depend on the availability of their substitutes. The arqument of the supply schedule is such that $S(p)$ is dependent on the captured fish. Therefore, $S(p) = qnE$. With the functions of $S(p)$ and $D(p)$ thus defined, the equation in market price becomes

$$\dot{p} = \alpha p(A - p - qnE). \tag{5}$$

Equation (5), has market price evolving nonlinearly depending on the price dynamics and the difference between supply and demand. The existence of price dynamics is occasioned by the price fluctuations on the market due to market forces caused by supply variations. This accounts for the nonlinearity in the price equation. This variation is more realistic than linear price variation considered in [7] and Auger *et al,* [2].

Equations (2), (4), and (5) make the simple time continous model in Equation(1) to become

$$\begin{aligned}
\dot{n} &= rn(1-\frac{n}{k}) - qnE, \\
\dot{E} &= \beta E(pqn - c), \\
\dot{p} &= \alpha p(A - p - qnE).
\end{aligned} \tag{6}$$

The analysis of the system of equations in (6) is the core concern of this paper.

We now outline how we shall study the model in Equation (6). In §2, we simplify the model by aggregating variables to reduce its dimension then obtain the equilibrium points of the reduced system and analyse their local stability in order to study its long term solutions. §3 is dedicated to Bifurcation analysis while the §4 and §5 are devoted to physical interpretation of the results and the conclusion respectively.

2 Long Term Solutions

The rate of change of p in (6) is comparatively faster than the evolution of the fish stock and fishing effort, this is due to the day to day variation of market price as the suppliers adjust to market forces and fishery conditions in order to recoup their investment and make profit. We reduce the model in Equation (6), to a two dimensional system by an appropriate aggregation of variables, see for instance [11] for more on simplification and scaling and Auger *et.al,* [2].

In our aggregation we replace p in the harvesting equation with its non trivial equilibrium values, $p := p^*$ which solves

$$\dot{p} = \alpha p(A - p - qnE) = 0, \tag{7}$$

to obtain $p^* = A - qnE$. Using (7) and p^* in Equation(6), we obtain

$$\begin{aligned}
\dot{n} &= n(r(1-\frac{n}{k}) - qE), \\
\dot{E} &= \beta E(-c + qn(A - qnE)),
\end{aligned} \tag{8}$$

a system of two equations that is relatively easy to analyse. We take $\beta = 1$ in (8), which is a maximum value in the range $0 \leq \beta \leq 1$ and may occur when the environmental conditions and harvesting are favourable for stock growth in the fishery.

Local Stability Analysis . In Equation(8), the n nullclines are: $n = 0$ and $r(1-\frac{n}{k})-qE = 0$ while the E nullclines are: $E = 0$ and $-c+qn(A-qnE) = 0$. The equilibrium points are the intersection of E and n nullclines; that is, $E_0 := (0, 0), E_1 := (k, 0)$ and $E_2 := (n^*, E^*)$ that is a solution (n^*, E^*) of

$$E(n) = \frac{r}{q}(1 - \frac{n}{k}), \quad E(n) = \frac{1}{qn}(A - \frac{c}{qn}). \tag{9}$$

Solving for c in Equation (9), we obtain a cubic equation for the parameter c as a function of the equilibrium fish stock $c(n^*)$ given by

$$c(n^*) := \frac{rq}{k} n^{*3} - rqn^{*2} + Aqn^*. \tag{10}$$

Solving Equation (10) with different values of k, we obtain one or three equilibrium values n^*.

It can easily be shown that E_0 is a saddle point. The point E_1, is stable if $k < \frac{c}{Aq}$ and is a saddle if $k > \frac{c}{Aq}$.

Linearization of Equation(8) at the equilibrium point E_2, gives the Jacobian matrix

$$J(n^*, E^*) = \begin{pmatrix} -\frac{r}{k}n^* & -qn^* \\ qE^*(A - 2qn^*E^*) & -q^2n^{*2}E^* \end{pmatrix}.$$

The trace, $tr(J)$, and the determinant, $det(J)$, of $J(n^*, E^*)$ are determined to establish the nature of its eigenvalues.They are given thus

$$tr(J) := -\frac{r}{k}n^* - q^2n^{*2}E^* < 0 \quad \text{and} \quad det(J) := q^2n^*E^*(\frac{r}{k}n^{*2} + A - 2qn^*E^*).$$

Using the first equation in Equation (9); that is, $E^* = \frac{r}{q}(1 - \frac{n^*}{k})$, we obtain $det(J) = q^2n^*E^*\psi(n^*)$, where $\psi(n^*) := \frac{3r}{k}n^{*2} - 2rn^* + A$. Since $q^2n^*E^*$ is positive, the sign of $det(J)$ will consequently depend on $\psi(n^*)$.

Since $c'(n) = \psi(n)$, where the prime indicates differentiation with respect to n, we study the sign of $\psi(n^*)$ by analysing $c(n^*)$ defined in Equation (10). First and foremost, let us consider Figure 1, not drawn to scale, that shows how $c(n^*)$ depends on k. The function $c(n^*)$ is plotted against n for values of $k = 2, 3, 4$ and 5.

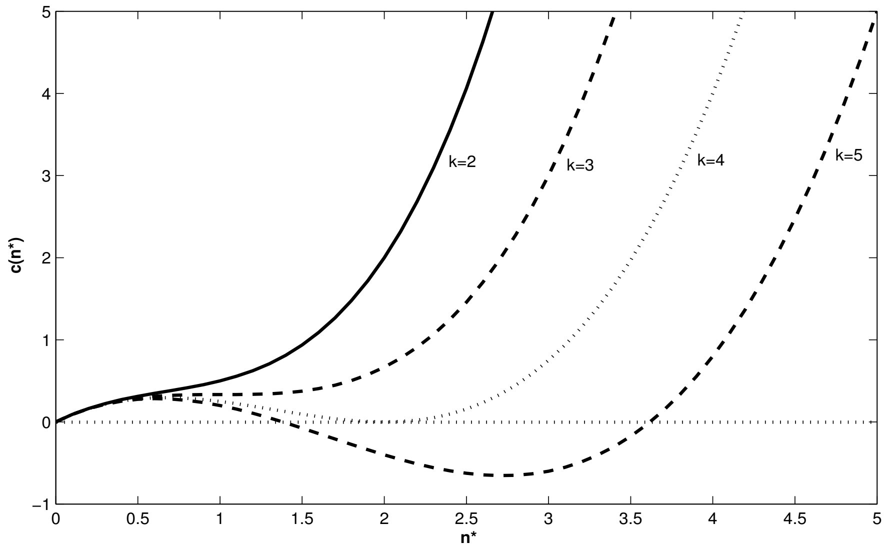

Figure 1: The graph of the function $c(n^*)$ for four values of k.

Figure 1 enables us to see how the number of zeroes vary with k and hence simplify the analysis of $c\prime(n^*)$. There are special points of $c(n^*)$ where two zeroes of $c(n^*)$ merge, this happens when $c\prime(n^*) = \frac{3rq}{k}n^{*2} - 2rqn^* + Aq = q\psi(n^*) = 0$.

The solutions n^* for $c\prime(n^*) = 0$ are

$$n^*_{1,2} = \frac{k}{3}\left(1 \pm \sqrt{1 - \frac{3A}{kr}}\right). \tag{11}$$

If $r < \frac{3A}{k}$, then $c\prime(n^*)$ is positive and $c(n^*)$ is monotonic increasing with complex roots. If $r > \frac{3A}{k}$ then there are two real zero's for $c\prime(n^*)$. If $r = \frac{3A}{k}$, the two real zero's coincide. Further analysis contained in the following propositions and their proofs distinguishes two different cases.

Proposition 2.1. *For $0 < r < \frac{3A}{k}$ and $k > \frac{c}{Aq}$, E_1 is a saddle point and E_2 is a positive stable equilibrium point*

Proof. Proposition 2.1.

If $0 < r < \frac{3A}{k}$ in (11), the sign of $c\prime(n^*)$ which is the same as the sign of $\psi(n^*)$ does not change and is always positive. This implies that $det(J) > 0$. Moreover, $c\prime\prime(n^*) = \frac{6rq}{k}n^* - 2rq$ implies that $n^* = \frac{k}{3}$ is a point of inflection. We have $c(k) = qAk$ but since c is strictly increasing and may take positive or negative values depending on k, we consider $c(k) = qAk - c$ and as $\lim_{n\to+\infty} c(n) = +\infty$, we conclude that c vanishes at a unique point n^*, thus we obtain a unique equilibrium point E_2. If $k < \frac{c}{Aq}$, then $c(k) < 0$ and c vanishes at a value $n^* > k$, which corresponds to a negative effort equilibrium ($E^* < 0$). In this case, the equilibrium point E_1 is a stable equilibrium but

(n^*, E^*) is of no interest since it is corresponding to unrealistic negative fishing effort, but if $k > \frac{c}{Aq}$ then $c(k) > 0$ and c vanishes at a value $n^* < k$, with a positive effort equilibrium $E^* > 0$. In this case E_1 is a saddle point and E_2 is the unique positive stable as $tr(J) < 0$ and $det(J) > 0$. □

Proposition 2.2. *For $0 < \frac{3A}{k} < r$, $E_i := (n_i^*, E_i^*)$ for $i = 1, 2, 3$ are three positive equilibrium points such that we have the following cases:*

1. *If $c(n^*) < 0$, $c(n_1^*) < 0$, we obtain a unique positive and stable equilibrium point (n^*, E^*);*
2. *If $c(n_1^*) > 0$ and $c(n_2^*) > 0$, we obtain a unique positive and stable equilibrium point (n^*, E^*);*
3. *If $c(n_1^*) > 0$ and $c(n_2^*) < 0$, we obtain three positive equilibrium points (n_i^*, E_i^*) for $i = 1, 2, 3$ whereby (n_1^*, E_1^*) and (n_3^*, E_3^*) are stable while (n_2^*, E_2^*) is a saddle equilibrium point.*

Proof. Proposition 2.2

If $0 < \frac{3A}{k} < r$ in (11), $c\prime$ vanishes at two values n_1 and n_2 given by

$$0 \leq n_1 = \frac{k}{3}\left(1 - \sqrt{1 - \frac{3A}{rk}}\right) < \frac{k}{3}, \quad \text{and} \quad \frac{k}{3} < n_2 = \frac{k}{3}\left(1 + \sqrt{1 - \frac{3A}{rk}}\right) < k.$$

Since $det(J)$, $\psi(n)$ and $c\prime(n)$ have the same sign, $det(J) > 0$ if $n\epsilon[0, n_1) \bigcup (n_2, +\infty]$, and $det(J) < 0$ if $n\epsilon(n_1, n_2)$.

Recall that $\lim_{n^* \to +\infty} c(n^*) = +\infty$. As $c(n_1^*)$ and $c(n_2^*)$ can have positive or negative signs, so for Case 1, with $c(n^*) < 0$, $c(n_1^*) < 0$ and $n^* > n_1$, $det(J) > 0$ and $tr(J) < 0$ thus a stable equilibrium point (n^*, E^*). For Case 2, since $c(n_1^*) > 0$ and $c(n_2^*) > 0$, with $n^* < n_1^*$, $det(J) > 0$ and $tr(J) < 0$ thus a stable equilibrium point (n^*, E^*). Finally for Case 3, given that $n_1^* < n_1 < n_2^* < n_2 < n_3^*$ is satisfied, (n_1^*, E_1^*) and (n_3^*, E_3^*) are stable since $det(J) > 0$ and $tr(J) < 0$ while (n_2^*, E_2^*) is a saddle equilibrium point since $det(J) < 0$ and $tr(J) < 0$. □

3 Bifurcation Analysis

We express Equation(8) in dimensionless terms and present one-parameter bifurcation diagrams showing how the change in parameter affects the dynamics of the system.

We nondimensionalise Equation(8) by making the following transformations;

$$\upsilon := \sqrt{q}n, \varepsilon := \frac{\sqrt{q}}{A}E, \tau := \sqrt{q}At, \rho := \frac{r}{A\sqrt{q}}, \gamma := \frac{c}{A\sqrt{q}}, \kappa := \sqrt{q}k. \qquad (12)$$

Using (12) in (8), we obtain its dimensionless form thus

$$\begin{aligned} \dot{\upsilon} &= \upsilon(\rho(1-\frac{\upsilon}{\kappa})-\varepsilon), \\ \dot{\varepsilon} &= \varepsilon(-\gamma+\upsilon(1-\varepsilon\upsilon)), \end{aligned} \quad (13)$$

where the dot ″.″ denotes differentiation with respect to time τ. We are now left with three parameters; κ, ρ, γ.

Let us interprete these parameters. In case $\rho \ll 1$, and $\gamma \ll 1$, then $A\sqrt{q} \gg r$ and $A\sqrt{q} \gg c$ where we have demand driven over-exploitation of the resource. If $\gamma \gg 1$ and $\rho \gg 1$ then it follows that $A\sqrt{q} \ll r$ and $A\sqrt{q} \ll c$ which will lead to under-exploitation of the fish resource.

In Equation(8), we observe that if we set $r = q = A = 1$ in Equation(6), we obtain Equation(13) with $\upsilon = n$, $\varepsilon = E$, $\rho = r$, $\gamma = c$ and $\kappa = k$, thus, we use initial parameters k, c and r. This will show us the long- term dynamic behaviour of the aggregated model. We shall show that there is a value of the bifurcation parameter $k =: k_0$ where the system in (6)undergoes a saddle-node bifurcation showing the co-existence of two stable equilibria separated by a saddle, whereby the fish population and the fishing effort varies with k. This is done by stating and proofing Proposition 3.1 and describing two bifurcation diagrams that show the number and type of stability of points of equilibria as k is varied.

Proposition 3.1. *For $n > 2c$, there is a value of $k =: k_0$ where the system in Equation(9) undergoes saddle - node bifurcation as the fish population and the fishing effort dynamics varies with the carrying capacity. Furthermore, for $k < k_0$, there are only two equilibria while when $k > k_0$, two more equilibria emerge, one stable and the other unstable.*

Proof. Proposition 3.1

Using Equation(8) and (9) and further aggregation, we obtain

$$\dot{n} = n\phi(n,k) =: \Phi(n,k), \quad (14)$$

where $\phi(n,k) := 1 - \frac{n}{k} - \frac{1}{n} + \frac{c}{n^2}$. Clearly, $n = 0$ and the curve $\phi(n,k) = 0$ gives the equilibrium points. For the stability of the equilibria points $\phi(n,k) = 0$, we have

$$\Phi'(n,k) = \phi'(n,k).$$

There is stability when $\phi'(n,k) < 0$ and instability when $\phi'(n,k) > 0$.

Since $\phi'(n,k)$ is continuous for $n > 0$, we have a change in stability at $\phi'(n,k) = -n^3 - 2ck + nk = 0$, and find that

$$k = k_0 := \frac{n^3}{n-2c}, \quad (15)$$

as the value of k where a bifurcation occurs.

To be able to indicate the nature of stability in the bifurcation diagram in Figure 2 obtained using Equation(14), we observe that on the curve $\phi(n,k) = 0$,

$$\frac{d\phi}{dk} = \frac{n*}{k^2} \mid_{(n^*,k^*)} > 0.$$

Thus by the Implicit Function Theorem, there exists

$$\phi(n, k(n)) = 0, \tag{16}$$

$k(n)$ defined in the neighbourhood of (n^*, k^*) with $k(n^*) = k^*$ as smooth as $\phi(n,k)$. Differentiating (16) with respect to n, we get

$$\frac{d\phi}{dn} = -\frac{d\phi}{dk}\frac{dk}{dn},$$

from which we can see that $sign(\frac{d\phi}{dn}) = -sign(\frac{dk}{dn})$. Hence the nature of stability in the bifurcation diagram Figure 2, where the variation of k, beyond $k := k_0$ leads to the creation of two additional equilibrium solutions.

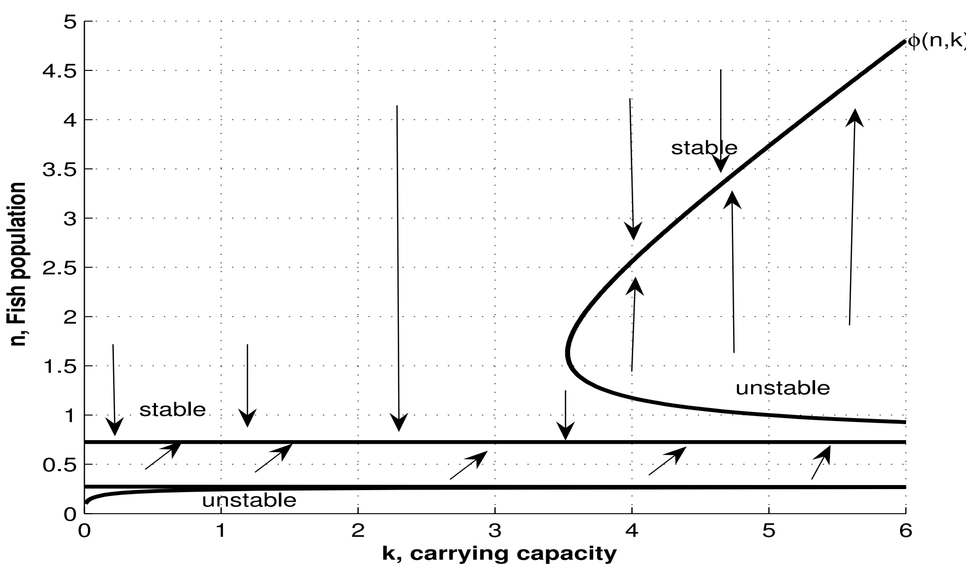

Figure 2: One- parameter bifurcation diagram for fish population with k as the bifurcation parameter

For the variation of the fishing effort with the carrying capacity as the bifurcation parameter, we obtain

$$\dot{E} = E(-c + n - n^2 E),$$

where we have set $q = A = 1$ in Equation(8). Similarly, we also obtain

$$n = k(1 - E),$$

from Equation (9) and hence

$$k = k_0 := \frac{k^3(1-E)^3}{k(1-E)-2c},$$

is the bifurcation value and further aggregation yields

$$\dot{E} = E\Theta(E,k) =: \Psi(E,k), \tag{17}$$

where $\Theta(E,k) := -c + k - kE - k^2E + 2k^2E^2 - k^2E^3$. Clearly, $E = 0$ and the curve $\Theta(E,k) = 0$ gives equilibrium points. For the stability of the equilibrium points $\Theta(E,k) = 0$, we have

$$\Psi'(E,k) = E\Theta'(E,k),$$

where the prime indicates differentiation with respect to E. There is stability when $\Theta'(E,k) < 0$ and instability when $\Theta'(E,k) > 0$. Since $\Theta'(E,k)$ is continuous with $E > 0$, there is a change in stability at $\Theta'(E,k) = 0$. To indicate the nature of stability, consider

$$\dot{E} = E\Theta(E,k) = 0.$$

$$\frac{d\Theta}{dk} = 1 - E - 2kE + 4kE^2 - 2kE^3,$$

where it is seen that $\frac{d\Theta}{dk} |_{(E^*,k^*)} < 0$ for $E^* > 0$. Thus, by the Implicit Function Theorem, there exists

$$\Theta(E,k(E)) = 0, \tag{18}$$

$k(E)$ defined in the neighbourhood of (E^*,k^*) with $k(E^*) = k^*$ as smooth as $\Theta(E,k)$. Differentiating Equation(18) with respect to E, we obtain

$$\frac{d\Theta}{dE} = -\frac{d\Theta}{dk}\frac{dk}{dE}.$$

Since $\frac{d\Theta}{dk} < 0$, it is evident that $sign(\frac{d\Theta}{dE}) = sign(\frac{dk}{dE})$, as seen in Figure 3 that is obtained from Equation (18). The stability changes at $\Theta'(E,k) = 0$, hence the nature of stability shown.

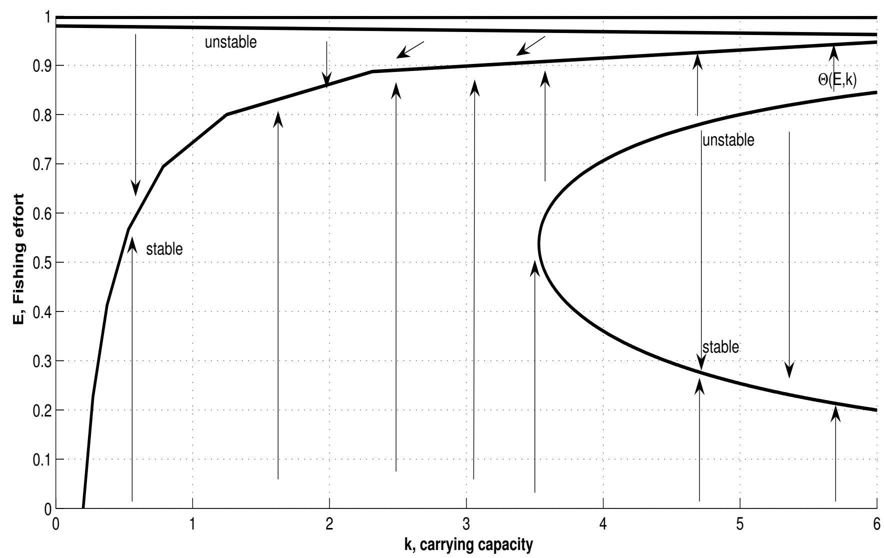

Figure 3 One- parameter bifurcation diagram for fishing effort with k as the bifurcation parameter

□

4 Comparison of the Fish Price in the case of two Stable Positive Equilibria

In this section we consider the variation of the equilibrium price in the case of the co-existence of two stable equilibria found in the local stability and bifurcation analysis. This portrays the effects of the nonlinear price variation on the dynamics of the fishery.

If $r > \frac{3A}{k}$, the three equilibria $(n_1^*, E_1^*), (n_2^*, E_2^*)$ and (n_3^*, E_3^*) are in the positive quadrant with (n_2^*, E_2^*) being a saddle while the other two equilibria are stable. Assume $n_3^* > n_1^*$, we have

$$E_1^* = \frac{1}{qn_1^*}\left(A - \frac{c}{qn_1^*}\right), \qquad E_3^* = \frac{1}{qn_3^*}\left(A - \frac{c}{qn_3^*}\right) \tag{19}$$

and

$$p_1^* = A - qn_1^*E_1^*, \qquad p_3^* = A - qn_3^*E_3^*. \tag{20}$$

Combining the two sets of expressions in (19) and (20), we obtain:

$$p_1^* - p_3^* = \frac{c}{q}\frac{(n_3^* - n_1^*)}{n_1^* n_3^*}. \tag{21}$$

The sign of the difference of price at equilibrium is opposite to the difference of the fish population. Thus, if $n_3^* > n_1^*$, then we have $p_1^* > p_3^*$. In general we have the following set of inequalities:

$$n_3^* > n_1^*, \quad E_3^* < E_1^*, \quad p_3^* < p_1^*.$$

Explained thus; at equilibrium, the larger the fish stock the smaller the fishing effort and the smaller is the market price, see Figure 4.

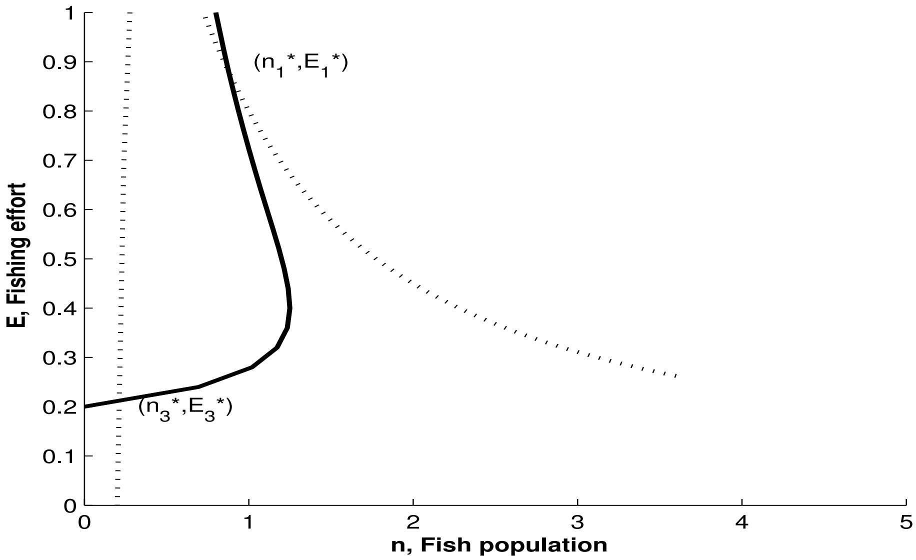

Figure 4: A phase-plane plot of fish population against fishing effort with $r = 1, c = 0.2, k = 4$

The solid line for the fishing effort intersects the dashed line for fish population at two points (n^*, E^*), such that any initial condition situated above these points leads to over-exploitation of the fish while any other initial condition below it leads to under-exploitation. Thus, this model predicts that we can have two different kinds of fisheries co-existing, namely:

1. **An over-exploited fishery** (n_1^*, E_1^*). Permitting large fishing effort and an economic activity with a satisfying market price p^*. However, the resource is maintained at a low level and due to some other factors, there may be a risk of fish extinction.

2. **An under-exploited fishery** (n_3^*, E_3^*). The fishery maintains the fish stock at a desirable large level which is far from extinction but any important economic activity is not supported. A small fishing effort with a relatively small market price is exhibited.

5 Conclusion

The results have shown that taking into account the variation of the market price induces three different scenarios of equilibria. The parameter values of k

and c, determine wheather we get one, two or three strictly positive equilibria co-existing.

In the case $E_0 = (0, 0)$, the absence of fish population implies that no fishing can take place and if there is introduction of a fish population, the fish population increases due to natural growth which will attract fishing activity. Whereas, if the fish population is diminishing due to environmental conditions or over- exploitation the fishing effort will also decline, thus this state is very unstable. For $E_1 = (k, 0)$, the fish population is maintained at its carrying capacity and with the absence of harvesting, the fishery persists at the carrying capacity hence stable equilibrium condition.

The case in which the two strictly stable equilibria exist shows that the two different stable equilibria can co-exist for the same fishery, this is due to taking into account of the varying market price. In each case, the two situations have advantages and disadvantages namely:

1 In case of over-exploitation, the fishery supports a large economic activity but the stock risks extinction.

2 In the case of under-exploited fishery, the stock is maintained at a large level, far from extinction but the fishery only supports a small economic activity.

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