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# On Summing Multipliers and Applications

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A scalar sequence $(\alpha_i)$ is said to be a $p$-summing multiplier of a Banach space $E$, if $\sum_{i=1}^{\infty}\|\alpha_i x_i\|^p < \infty$ for all weakly $p$-summable sequences in $E$. We study some important properties of the space $m_p(E)$ of all $p$-summing multipliers of $E$, consider applications to $E$-valued operators on the sequence space $l^p$, and extend this work to general "summing multipliers." The case $p = 1$ shows close resemblance to the work of B. Marchena and C. Piñeiro (*Quaestiones Math*., to appear), where the results originated from the authors' interest in sequences in the ranges of vector measures. © 2001 Academic Press

## INTRODUCTION AND NOTATION

Let $E$, $F$ be Banach spaces over $\mathbb{K} \in \{\mathbb{C}, \mathbb{R}\}$. $L(E,F)$ denotes the space of all bounded linear operators between $E$ and $F$, whereas $K(E,F)$ denotes the space of all compact linear operators between $E$ and $F$. $L(E,\mathbb{K})$ is denoted by $E^*$ and when convenient we use the notation $\langle x, x^*\rangle$ for $x^*(x)$ where $x \in E$, $x^* \in E^*$. The closed unit ball in $E$ will be denoted $B_E$. Sequences in $E$ will be denoted $(x_i)$, $(y_i)$, etc., and we let

$$(x_i)(\leq n) := (x_1, x_2, \dots, x_n, 0, 0, \dots),$$
$$(x_i)(\geq n) := (0, 0, \dots, 0, x_n, x_{n+1}, \dots).$$

A vector space $\Lambda$ whose elements are sequences $(\alpha_n)$ of numbers (real or complex) is called a sequence space. $\Lambda$ is said to be normal if whenever it contains $(\alpha_n)$, it also contains all sequences $(\beta_n)$ with $|\beta_n| \leq |\alpha_n|$ for all

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$n \in \mathbb{N}$. To each sequence space $\Lambda$ we assign another sequence space $\Lambda^{\times}$, its Köthe-dual, which is the set of all sequences $(\beta_n)$ for which the series $\sum_{n=1}^{\infty} \alpha_n \beta_n$ converge absolutely for all $(\alpha_n) \in \Lambda$.

A sequence space $\Lambda$ is said to be *perfect* if $\Lambda^{\times\times} = \Lambda$. $\Lambda$ is said to be *symmetric* if $(\alpha_i) \in \Lambda$ if and only if $(\alpha_{\pi(i)}) \in \Lambda$ for all permutations $\pi$ of the positive integers.

A Banach sequence space $\Lambda$ is said to be a *BK-space* if each coordinate projection mapping $(\alpha_n) \mapsto \alpha_i$ is continuous.

Let $e_n = (\delta_{i,n})_i$, with $\delta_{i,n} = 1$ if $i = n$ and $\delta_{i,n} = 0$ if $i \neq n$. In a dual normed sequence space $\Lambda^*$ we will use the notation $e_n^*$ for $e_n$. A normed sequence space is said to have the *AK-property* if all its elements can be approximated by their sections, that is, if each element $(\beta_i)$ in the sequence space satisfies $(\beta_i) = \lim_{n \to \infty} (\beta_i)(\leq n)$, where $(\beta_i)(\leq n) = \sum_{i=1}^{n} \beta_i e_i$. A *BK*-space $\Lambda$ has the *AK*-property if and only if $\{e_n : n = 1, 2, \ldots\}$ is a Schauder basis for $\Lambda$, that is, if and only if $\lim_{n \to \infty} \|(\mu_i)(\geq n))\|_{\Lambda} = 0$. If $\Lambda$ is a normal *BK*-space with *AK*, then $\{e_n : n = 1, 2, \ldots\}$ is an unconditional basis for $\Lambda$, called the *standard coordinate basis* or the unit vector basis of $\Lambda$. In this case a standard argument shows that $\Lambda^{\times}$ is algebraically isomorphic to the continuous dual space $\Lambda^*$ with respect to the obvious duality. We call $\Lambda$ a *DAK* space if both $\Lambda$ and $\Lambda^{\times}$ have the *AK* property.

If not otherwise stated, all scalar sequence spaces $\Lambda \neq l^{\infty}$ will throughout be assumed to be normal symmetric *BK*-spaces with the *AK*-property. In this case we may assume that $\|e_n\|_{\Lambda} = 1$ for all $n \in \mathbb{N}$. For information on scalar sequence spaces we refer to [11].

The following standard sequence spaces will be referred to:

- $w$, the vector space (with respect to coordinate wise vector operations) of all (complex and real) scalar sequences;
- $\phi \subset w$, the space of all sequences with only a finite number of non-zero terms;
- $l^{\infty}$, the space of all bounded sequences;
- $c_0$, the space of all null sequences;
- $l^p$, $1 \leq p < \infty$, the space of all absolutely $p$-summable sequences.

The vector sequence space $\Lambda_s(E) := \{(x_i) \subset E : (\|x_i\|) \in \Lambda\}$ is a complete normed space with respect to the norm

$$\pi_{\Lambda}((x_i)) := \|(\|x_i\|)\|_{\Lambda}.$$

We put $\pi_{\Lambda}((x_i)) = \pi_p((x_i))$ when $\Lambda = l^p$, the Banach space of $p$-absolutely summable scalar sequences (with $1 \leq p < \infty$).

The vector sequence space $\Lambda_w(E) := \{(x_i) \subset E : (\langle x_i, a \rangle) \in \Lambda, \forall a \in E^*\}$ is a complete normed space with respect to the norm

$$\epsilon_\Lambda((x_i)) := \sup_{\|a\| \leq 1} \|(\langle x_i, a \rangle)\|_\Lambda.$$

We put $\epsilon_p = \epsilon_\Lambda$ when $\Lambda = l^p$ (with $1 \leq p < \infty$).

The vector sequence space

$$\begin{aligned} \Lambda_c(E) &= \left\{ (x_i) \in \Lambda_w(E) : (x_i) = \epsilon_\Lambda - \lim_{n \to \infty} (x_1, \dots, x_n, 0, \dots) \right\} \\ &= \left\{ (x_i) \in \Lambda_w(E) : \epsilon_\Lambda((x_i)(\geq n)) \to 0 \text{ if } n \to \infty \right\} \end{aligned}$$

is a closed subspace of $\Lambda_w(E)$. On $\Lambda_c(E)$ we will consider the induced subspace norm, inherited from $\Lambda_w(E)$.

The vector sequence space $\Lambda_w(E^*) := \{(x_i^*) \subset E^* : (\langle x, x_i^* \rangle) \in \Lambda, \forall x \in E\}$ is a complete normed space with respect to the norm

$$\epsilon_\Lambda((x_i^*)) := \sup_{\|x\| \leq 1} \|(\langle x, x_i^* \rangle)\|_\Lambda.$$

We put $\epsilon_p = \epsilon_\Lambda$ when $\Lambda = l^p$ (with $1 \leq p < \infty$).

The vector sequence space

$$\begin{aligned} \Lambda_c(E^*) &= \left\{ (x_i^*) \in \Lambda_w(E^*) : (x_i^*) = \epsilon_\Lambda - \lim_{n \to \infty} (x_1^*, \dots, x_n^*, 0, \dots) \right\} \\ &= \left\{ (x_i^*) \in \Lambda_w(E^*) : \epsilon_\Lambda((x_i^*)(\geq n)) \to 0 \text{ if } n \to \infty \right\} \end{aligned}$$

is a closed subspace of $\Lambda_w(E^*)$. On $\Lambda_c(E^*)$ we will consider the induced subspace norm, inherited from $\Lambda_w(E^*)$.

It follows from [9, Proposition 2] that the continuous dual space $\Lambda_c(E)^*$ can be identified with the vector space of all sequences $(x_i^*)$ in $E^*$ such that $\sum_{i=1}^\infty |\langle x_i, x_i^* \rangle| < \infty$ for all $(x_i) \in \Lambda_w(E)$.

It is proved in [6] that $(x_i) \in \Lambda_w^\times(E)$ if and only if $\sum_{i=1}^\infty \lambda_i x_i$ converges in $E$ for every $(\lambda_i) \in \Lambda$ and that

$$\epsilon_{\Lambda^\times}((x_i)) = \sup_{(\lambda_i) \in B_\Lambda} \left\| \sum_{i=1}^\infty \lambda_i x_i \right\|.$$

Moreover, the following characterisations can also be found in [6, 8]:

THEOREM 0.1. *Consider a Banach space $E$.*

(a) *Let $\Lambda$ be a Banach sequence space with the AK-property. Then $\Lambda_w^\times(E)$ is isometrically isomorphic to $L(\Lambda, E)$. The isometry is given by $(x_n) \mapsto T_{(x_n)}$, where $T_{(x_n)}((\xi_i)) = \sum_{i=1}^\infty \xi_i x_i$.*

(b) *Let* $\Lambda$ *be a Banach sequence space with the AK-property such that* $\Lambda^\times$ *has AK. Then* $\Lambda_c^\times(E)$ *is isometrically isomorphic to* $K(\Lambda, E)$. *The isometry is defined as in* (a).

(c) *Let* $\Lambda$ *be a Banach sequence space with the AK-property. Then* $\Lambda_w(E^*)$ *is isometrically isomorphic to* $L(E, \Lambda)$. *The isometry is given by* $(x_n^*) \mapsto T_{(x_n^*)}$, *where* $T_{(x_n^*)}x = (\langle x, x_n^* \rangle)$.

(d) *Let* $\Lambda$ *be a Banach sequence space with the AK-property. Then* $\Lambda_c^\times(E^*)$ *is isometrically isomorphic to* $K(E, \Lambda)$. *The isometry is defined as in* (c).

The following well known normed operator ideals will be considered in this work:

• $(\mathscr{F}, \|\cdot\|)$, where $T \in \mathscr{F}(X, Y)$ if and only if $T$ is a finite rank bounded linear operator and $\|\cdot\|$ is the usual uniform operator norm. Recall that $T \in \mathscr{F}(X, Y)$ if and only if $T$ has a representation of the form $T = \sum_{i=1}^n a_i \otimes x_i$ where $a_i \in X^*$ and $x_i \in Y$. Also, recall that the *trace* of $S = \sum_{i=1}^n x_i^* \otimes x_i \in \mathscr{F}(X, X)$ is the number

$$\operatorname{tr}(S) = \sum_{i=1}^n \langle x_i, x_i^* \rangle,$$

which is independent of the representation of $S$.

• $(\mathscr{N}, \nu_1)$, where $T \in \mathscr{N}(X, Y)$ if and only if $T$ is a nuclear operator; i.e., $T$ has a representation

$$Tx = \sum_{i=1}^\infty \lambda_i \langle x, x_i^* \rangle y_i,$$

where $(\lambda_i) \in l^1$, $(x_i^*)$ is bounded in $X^*$, and $(y_i)$ is bounded in $Y$. Here

$$\nu_1(T) := \inf \sum_{i=1}^\infty |\lambda_i|,$$

where the infimum is extended over all such representations for which $\|x_i^*\| \le 1$ and $\|y_i\| \le 1$ for all $i$.

• $(\mathscr{I}_1, i_1)$, where $T \in \mathscr{I}_1(X, Y)$ if and only if $T$ is an integral operator, i.e., if and only if there exists $\rho \ge 0$ such that

$$|\operatorname{tr}(TS)| \le \rho \|S\|, \qquad \forall S \in \mathscr{F}(Y, X).$$

The integral norm $i_1(T)$ equals the smallest of all numbers $\rho \ge 0$ admissible in these inequalities.

• $(\mathscr{P}_p, \pi_p)$, where $T \in \mathscr{P}_p(X,Y)$ if and only if $T$ is a $p$-absolutely summing operator, i.e., if and only if $(Tx_i) \in l_s^p(Y)$ for all $(x_i) \in l_w^p(X)$. The $p$-summing norm $\pi_p(T)$ of $T$ equals the operator norm of the bounded linear operator

$$l_w^p(X) \to l_s^p(Y) :: (x_i) \mapsto (Tx_i);$$

i.e.,

$$\pi_p(T) = \sup\left\{\left(\sum_{i=1}^{\infty} \|Tx_i\|^p\right)^{1/p} : \epsilon_p((x_i)) \leq 1\right\}.$$

We recall some information in connection with vector measures.

Definition 0.2. A function $G$ from a field $\Sigma$ of subsets of a set $\Omega$ to a Banach space $X$ is called a finitely additive vector measure if whenever $A_1$ and $A_2$ are disjoint members of $\Sigma$ then

$$G(A_1 \cup A_2) = G(A_1) + G(A_2).$$

If in addition

$$G\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} G(A_n)$$

in the norm topology of $X$ for all sequences $(A_n)$ of pairwise disjoint members of $\Sigma$ such that $\bigcup_{n=1}^{\infty} A_n \in \Sigma$, then $G$ is termed a countably additive vector measure.

Definition 0.3. Let $G$: $\Sigma \to X$ be a vector measure. The variation of $G$ is the extended non-negative function $|G|$ whose value on a set $A \in \Sigma$ is given by

$$|G|(A) = \sup_{\pi} \sum_{B \in \pi} \|F(B)\|,$$

where the supremum is taken over all partitions $\pi$ of $A$ into a finite number of pairwise disjoint members of $\Sigma$. If the total variation of $G$ (that is, $|G|(\Omega)$) is finite, then $G$ is called a measure of bounded variation.

Definition 0.4. The range of a vector-valued measure is a set of the form

$$\mathrm{rg}(G) := \{G(A) : A \in \Sigma\},$$

where $\Sigma$ is a $\sigma$-field of sets and $G$ is a countably additive measure on $\Sigma$ with values in a Banach space.

The *Pietch integral operators* are defined as follows:

- $T \in \mathscr{P}g(X, Y)$ if and only if there exists a $Y$-valued countably additive vector measure $G$ of bounded variation defined on the Borel (for the weak*-topology) sets of the closed unit ball $B_{X^*}$ of $X^*$ such that for each $x \in X$ we have

$$T(x) := \int_{B_{X^*}} x^*(x)\, dG(x^*).$$

The space $\mathscr{P}g(X, Y)$ becomes a Banach space under the norm

$$\|T\|_{\text{pint}} = \inf\{|G|(B_{X^*})\},$$

where the infimum is taken over all measures $G$ that satisfy the above definition.

In Section 1 of the present paper we introduce the concept *p-summing multiplier* for a Banach space $E$. This is a scalar sequence $(\alpha_i)$ such that the coordinatewise products of $(\alpha_i)$ with the weak $p$-summable sequences in $E$ are absolutely $p$-summable. We define a natural norm on the scalar sequence space $m_p(E)$ of all $p$-summing multipliers of $E$ and prove some inclusion theorems (such as $m_p(E) \subseteq m_q(E)$ if $1 < p \leq q < \infty$). The inclusion $m_p(E^{**}) \subseteq m_p(E)$ is easy to verify; however, it follows from our discussion in Section 1 that indeed $m_p(E) = m_p(E^{**})$. Let $(\mu_i)$ be a bounded scalar sequence. Our discussion in Section 1 also shows that the operators $l^p \to E$: $(\beta_i) \mapsto \sum_{i=1}^{\infty} \beta_i \mu_i x_i$ (with $\frac{1}{p} + \frac{1}{q} = 1$) are integral (or nuclear) for all absolutely $q$-summable sequences (with $1 < q < \infty$) if and only if $(\mu_i) \in m_p(E^*)$.

In Section 2 we consider the relation of $m_1(E)$ (which is denoted by $m(E)$ and whose elements are called *absolutely summing multipliers* of $E$) with the Orlicz property, find $m(E)$ for infinite dimensional $L^p$-spaces, and discuss the connection of our work on absolutely summing multipliers with work in [12]. It is shown that for a given bounded scalar sequence $(\mu_i)$ and all norm null sequences $(x_i^*) \subset E^*$, the operators

$$l^1 \to E^*: (\beta_i) \mapsto \sum_{i=1}^{\infty} \beta_i \mu_i x_i^*$$

are nuclear if and only if $(\mu_i) \in m(E)$.

The work of Section 1 is extended in Section 3 to obtain results for so called $(\Lambda, \Sigma)$-*summing multipliers*. The last part of Section 3 is devoted to a study of properties on the underlying sequence and Banach spaces that will ensure that the Banach sequence space $m_{\Lambda,\Sigma}(E)$ of $(\Lambda, \Sigma)$-summing multipliers has the *AK*-property. A general necessary and sufficient condition for $m_{\Lambda,\Sigma}(E)$ to have the $AK$-property is proved. From this general condition it is for instance possible to conclude that $m_{\Lambda^\times,\Sigma}(E)$ has $AK$ if both $E$ and $\Lambda$ are reflexive and $L(\Lambda, E) = K(\Lambda, E)$.

## 1. $p$-SUMMING MULTIPLIERS

DEFINITION 1.1. Let $1 \leq p \leq \infty$. A scalar sequence $(\alpha_i)$ is called a $p$-summing multiplier for a Banach space $X$, if $\sum_{n=1}^{\infty} \|\alpha_n x_n\|^p < \infty$ for all sequences $(x_n) \in l_w^p(X)$. Put

$$m_p(X) = \left\{ (\alpha_n) \in \omega : \sum_{n=1}^{\infty} \|\alpha_n x_n\|^p < \infty, \forall (x_n) \in l_w^p(X) \right\}.$$

$m_p(X)$ is a subspace of $l^\infty$. Since each $(x_i) \in l_w^p(X)$ is a norm bounded sequence in $X$, it is also clear that $l^p \subseteq m_p(X)$.

On the vector space $m_p(X)$ we define a norm

$$\|(\alpha_i)\|_{p,p} := \sup_{\epsilon_p((x_i)) \leq 1} \left( \sum_{n=1}^{\infty} |\alpha_n|^p \|x_n\|^p \right)^{1/p},$$

which is well defined because for each $(\alpha_i) \in m_p(X)$ this is the operator norm of the bounded (having closed graph) linear operator

$$T_\alpha : l_w^p(X) \to l_s^p(X) :: (x_i) \mapsto (\alpha_i x_i).$$

$m_p(X)$ is a complete normed space with respect to the above operator norm.

We first prove an inclusion relation between the different $p$-summing multiplier spaces of a fixed Banach space.

THEOREM 1.2. *If* $1 \leq p \leq q < \infty$ *then* $m_p(X) \subseteq m_q(X)$. *Moreover, if* $(\alpha_i) \in m_p(X)$, *then* $\|(\alpha_i)\|_{q,q} \leq \|(\alpha_i)\|_{p,p}$.

*Proof.* Let $p < q$ and fix $(\alpha_n) \in m_p(X)$. For an arbitrary $(x_i) \in l_w^q(X)$, put

$$\lambda_i = |\alpha_i|^{(q/p)-1} \|x_i\|^{(q/p)-1}.$$

Since $\frac{1}{q/p} + \frac{1}{q/(q-p)} = \frac{p}{q} + \frac{q-p}{q} = \frac{q}{q} = 1$, we have

$$\begin{aligned}
\left( \sum_{i=1}^{n} |\alpha_i|^q \|x_i\|^q \right)^{1/p}
&= \left( \sum_{i=1}^{n} |\alpha_i|^p \|\lambda_i x_i\|^p \right)^{1/p} \\
&\le \|(\alpha_i)\|_{p,p} \sup_{\|x^*\| \le 1} \left[ \left( \sum_{i=1}^{n} (\lambda_i^p)^{q/(q-p)} \right)^{(q-p)/q} \right. \\
&\qquad \left. \times \left( \sum_{i=1}^{n} \left( |\langle x_i, x^* \rangle|^p \right)^{q/p} \right)^{p/q} \right]^{1/p} \\
&= \|(\alpha_i)\|_{p,p} \sup_{\|x^*\| \le 1} \left( \sum_{i=1}^{n} |\langle x_i, x^* \rangle|^q \right)^{1/q} \left( \sum_{i=1}^{n} \lambda_i^{pq/(q-p)} \right)^{(q-p)/pq} \\
&= \|(\alpha_i)\|_{p,p} \epsilon_q\big((x_i)(\le n)\big) \left( \sum_{i=1}^{n} |\alpha_i|^q \|x_i\|^q \right)^{(1/p)-(1/q)}.
\end{aligned}$$

Thus it follows that

$$\begin{aligned}
\left( \sum_{i=1}^{n} |\alpha_i|^q \|x_i\|^q \right)^{1/q} &\le \|(\alpha_i)\|_{p,p} \epsilon_q\big((x_i)(\le n)\big) \\
&\le \|(\alpha_i)\|_{p,p} \epsilon_q\big((x_i)\big) \qquad \text{for all } n \in \mathbb{N}.
\end{aligned}$$

Therefore

$$\left( \sum_{i=1}^{\infty} |\alpha_i|^q \|x_i\|^q \right)^{1/q} \le \|(\alpha_i)\|_{p,p} \epsilon_q\big((x_i)\big) < \infty$$

for all $(x_i) \in l_w^q(X)$. Thus $(\alpha_i) \in m_q(X)$. The norm inequality for $(\alpha_i) \in m_p(X)$ is also clear from the last inequality. ∎

It is easy to verify the following

LEMMA 1.3. *If a Banach space $X$ is topologically isomorphic to a closed subspace of a Banach space $Y$, then*

$$m_p(Y) \subseteq m_p(X).$$

The proof of the following result is both a generalisation and adjustment of [12, proof of Theorem 1], where a similar result is proved for a sequence space $\lambda_X$ (which is in fact the same as the space $m_1(X^*)$—more about this will be said in Section 2).

THEOREM 1.4. *Let $(\alpha_i)$ be a bounded scalar sequence. Then $(\alpha_i) \in m_p(X^*)$ if and only if the operator $l^p \to X$: $(\beta_i) \mapsto \sum_{i=1}^{\infty} \beta_i \alpha_i x_i$ is integral for all sequences $(x_i) \in l_s^q(X)$. Here $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$.*

*Proof.* Let $(\alpha_i) \in m_p(X^*)$. Using $K(X, l^p) \cong l_c^p(X^*)$ (cf. Theorem 0.1), define $P$: $K(X, l^p) \to l_s^p(X^*)$ by

$$P\left(\sum_{n=1}^{\infty} x_n^* \otimes e_n\right) = (\alpha_n x_n^*).$$

Then $P$ is linear and bounded with

$$\left\|P\left(\sum_{n=1}^{\infty} x_n^* \otimes e_n\right)\right\| \leq \|(\alpha_n)\|_{p,p}\left\|\sum_{n=1}^{\infty} x_n^* \otimes e_n\right\|.$$

Since $l^p$ has the metric approximation property, the dual operator

$$P^*: l_s^q(X^{**}) \to K(X, l^p)^* = \mathscr{I}(l^p, X^{**}) \qquad (\text{cf. [10, p. 449]})$$

is bounded into the Banach space $\mathscr{I}(l^p, X^{**})$. Also,

$$P^*((x_n^{**})) = \sum_{n=1}^{\infty} e_n^* \otimes \alpha_n x_n^{**}.$$

Hence $\sum_n e_n^* \otimes \alpha_n x_n^{**} \in \mathscr{I}(l^p, X^{**})$ for all $(x_n^{**}) \in l_s^q(X^{**})$.

In particular $P^*((x_n)) = \sum_{n=1}^{\infty} e_n^* \otimes \alpha_n x_n \in \mathscr{I}(l^p, X^{**})$ for all $(x_n) \in l_s^q(X)$. However, $(\sum_n e_n^* \otimes \alpha_n x_n)(l^p) \subset X$; i.e.,

$$\sum_n e_n^* \otimes \alpha_n x_n \in \mathscr{I}(l^p, X), \qquad \forall (x_n) \in l_s^q(X) \qquad (\text{cf. [4, p. 233]}).$$

Conversely, let $(\alpha_i) \in l^\infty$ be given. Suppose the operator

$$l^p \to X: (\beta_i) \mapsto \sum_i \beta_i \alpha_i x_i$$

is integral for all $(x_i) \in l_s^q(X)$. Define

$$Q: l_s^q(X) \to \mathscr{I}(l^p, X) :: Q((x_n)) = \sum_n e_n^* \otimes \alpha_n x_n \in \mathscr{I}(l^p, X).$$

Since $l^q = (l^p)^*$ has the metric approximation property, it follows (cf. [10, p. 410]) that $\mathscr{N}(l^p, X)$ is isometric to a subspace of $\mathscr{I}(l^p, X)$. Now

$$Q((x_1, x_2, \dots, x_n, 0, 0, \dots)) = \sum_{i=1}^{n} e_i^* \otimes \alpha_i x_i$$

is a nuclear operator for all $n \in \mathbb{N}$ and $(x_i) \in l_s^q(X)$. From the continuity of $Q$ (having closed graph) and the fact that $(x_1, x_2, \dots, x_n, 0, 0, \dots) \overset{n}{\underset{\infty}{\to}} (x_i)$ in the norm of $l_s^q(X)$, it follows that $Q(l_s^q(X)) \subset \mathscr{N}(l^p, X)$. Hence $Q^*$: $L(X, l^p)(= \mathscr{N}(l^p, X)^*) \to l_s^p(X^*)$ is bounded.

Using $l_w^p(X^*) \overset{\text{isom}}{\cong} L(X, l^p)$, it follows via trace duality that

$$Q^*((x_n^*)) = (\alpha_n x_n^*) \qquad \text{for all } (x_i^*) \in l_w^p(X^*).$$

Therefore $\sum_n |\alpha_n|^p \|x_n^*\|^p < \infty$, $\forall (x_n^*) \in l_w^p(X^*)$; i.e. $(\alpha_n) \in m_p(X^*)$. ∎

A careful study of the last proof reveals that

THEOREM 1.5. *Let* $(\alpha_n) \in l^\infty$ *and* $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. *The following are equivalent*:

(a) $(\alpha_n) \in m_p(X^*)$.

(b) $\sum_n e_n^* \otimes \alpha_n x_n$: $l^p \to X$ *is integral for all* $(x_i) \in l_s^q(X)$.

(c) $\sum_n e_n^* \otimes \alpha_n x_n$: $l^p \to X$ *is nuclear for all* $(x_i) \in l_s^q(X)$.

COROLLARY 1.6. *Let* $(\alpha_n) \in l^\infty$ *and* $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. *The following are equivalent*:

(a) $(\alpha_n) \in m_p(X^{**})$.

(b) $\sum_n e_n^* \otimes \alpha_n x_n^*$: $l^p \to X^*$ *is integral for all* $(x_i^*) \in l_s^q(X^*)$.

(c) $\sum_n e_n^* \otimes \alpha_n x_n^*$: $l^p \to X^*$ *is nuclear for all* $(x_i^*) \in l_s^q(X^*)$.

LEMMA 1.7. *Let* $(\alpha_i) \in m_p(X)$ *and let* $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p, q < \infty$. *Consider the bounded linear operator*

$$P\colon K(l^q, X) \to l_s^p(X)\colon \sum_{n=1}^{\infty} e_n^* \otimes x_n \mapsto (\alpha_n x_n).$$

$P^*$ *maps* $l_s^q(X^*)$ *into* $\mathscr{N}(X, l^q)$ *and*

$$(P^*((x_i^*)))^* = \sum_{i=1}^{\infty} \alpha_i e_i^* \otimes x_i^* \in \mathscr{N}(l^p, X^*)$$

*for all* $(x_i^*) \in l_s^q(X^*)$.

*Proof.* The linear operator $P$ is clearly bounded, since

$$\pi_p((\alpha_i x_i)) \le \|(\alpha_i)\|_{p,p} \epsilon_p((x_i)).$$

Therefore

$$P^*\colon l_s^q(X^*) \to K(l^q, X)^* \cong \mathscr{I}(X, l^q)$$

is bounded. The isometry $K(l^q, X)^* \cong \mathscr{I}(X, l^q)$ is defined by trace duality (cf. [10, p. 449]). Also,

$$\left\langle P^*((x_i^*)), \sum_{n=1}^{\infty} e_n^* \otimes x_n \right\rangle = \langle (x_i^*), (\alpha_n x_n) \rangle = \sum_{i=1}^{\infty} \alpha_i x_i^*(x_i),$$

for all $(x_i) \in l_c^p(X)$. Fix $(x_i^*) \in l_s^q(X^*)$ and let

$$T_k\colon X \to l^q\colon x \mapsto (\alpha_n x_n^*(x))(\leq k).$$

For each $k \in \mathbb{N}$, $T_k$ is bounded and $T_k = \sum_{n=1}^k \alpha_n x_n^* \otimes e_n$. Now

$$\operatorname{tr}\left( T_k \circ \sum_{n=1}^{\infty} e_n^* \otimes x_n \right) = \sum_{j=1}^{k} \alpha_j x_j^*(x_j),$$

so that

$$\left\langle P^*((x_i^*)), \sum_{n=1}^{\infty} e_n^* \otimes x_n \right\rangle = \sum_{i=1}^{\infty} \alpha_i x_i^*(x_i) = \lim_k \left\langle T_k, \sum_{n=1}^{\infty} e_n^* \otimes x_n \right\rangle$$

for all $(x_i) \in l_c^p(X)$. This shows that for all $x \in X$, we have

$$P^*((x_i^*))(x) = \left( \sum_{n=1}^{\infty} \alpha_n x_n^* \otimes e_n \right)(x) \in l^q.$$

Since $l^q$ has the (metric) approximation property, $\mathscr{N}(X, l^q)$ is isometric to a subspace of $\mathscr{I}(X, l^q)$ (cf. [10, p. 410]). The continuity of $P^*$ thus implies that

$$\lim_n P^*((x_i^*)(\leq n)) = P^*((x_i^*)) \in \mathscr{N}(X, l^q),$$

whereby each $P^*((x_i^*)(\leq n)) = \sum_{i=1}^n \alpha_i x_i^* \otimes e_i$ is in $\mathscr{N}(X, l^q)$. The dual operator is also nuclear (cf. [10, p. 379]); thus

$$(P^*((x_i^*)))^* \in \mathscr{N}(l^p, X^*).$$

Moreover,

$$(P^*((x_i^*)))^* = \sum_{i=1}^{\infty} \alpha_i e_i^* \otimes x_i^*.$$

∎

It follows from Lemmas 1.3 and 1.7 and Corollary 1.6 that

PROPOSITION 1.8. *Let X be a Banach space and* $1 < p < \infty$. *Then*

$$m_p(X) = m_p(X^{**}).$$

## 2. THE SEQUENCE SPACE $m(E)$

In [7], Fourie used the name *absolutely summing multiplier* for what is, in the context of the previous section, a 1-summing multiplier. We therefore agree to henceforth refer to "absolutely summing multiplier" instead of "1-summing multiplier."

DEFINITION 2.1. A sequence $(\xi_i) \in \omega$ is called an absolutely summing multiplier of $E$ if $(\xi_i x_i)$ is absolutely summable in $E$ whenever $(x_i)$ is weakly absolutely summable in $E$; hence $(\xi_i x_i) \in l_s^1(E)$ for all $(x_i) \in l_w^1(E)$.

The scalar sequence space of all absolutely summing multipliers of $E$ is denoted by $m(E)$ (instead of $m_1(E)$, as would follow from the notation in Section 1). Clearly, if $E$ has finite dimension then $l_w^1(E) = l_s^1(E)$, so that $m(E) = l^\infty$. For all Banach spaces $E$, $m(E)$ is a perfect space (in the sense of Köthe) and if $E$ is infinite dimensional then $l^1 \subseteq m(E) \subseteq l^2$, the last inclusion following from the well known Dvoretzky–Rogers theorem (as is shown in [7]).

Recall that a sequence $(x_n) \subset E$ is said to be *unconditionally summable* if $\sum_n x_{\sigma(n)}$ converges in $E$, regardless of the permutation $\sigma$ of the indices. We refer to [3, Theorem 8, p. 45] for a proof of the fact that each $(x_i) \in l_w^1(E)$ is unconditionally summable if and only if $E$ does not contain a copy of $c_0$. There are many Banach spaces $E$ for which $m(E) = l^2$; actually it is easy to precisely characterise those spaces for which this property holds. Recall the *Orlicz property*:

DEFINITION 2.2 (cf. [3, p. 188]). A Banach space $E$ is said to have the Orlicz property if all unconditionally summable sequences in $E$ are in the space $l_s^2(E)$ of 2-absolutely summable sequences.

There are numerous examples of Banach spaces having the Orlicz property. For instance, by a result of B. Maury (cf. [3, p. 188]) this property characterises Banach lattices with cotype 2. All Banach spaces having cotype 2 have the Orlicz property. Hence if $E$ has type 2, then $E^*$ has the Orlicz property. Since the Banach lattice $c_0$ has no finite cotype, it is clear from the above discussion that $c_0$ does not have the Orlicz property. Hence it is a necessary condition for a Banach space with the Orlicz property not to contain a copy of $c_0$.

We are now ready to characterise the Banach spaces for which the space of absolutely summing multipliers is $l^2$.

THEOREM 2.3. *Let $E$ be an infinite dimensional Banach space. Then $m(E) = l^2$ if and only if $E$ has the Orlicz property.*

*Proof.* Let $m(E) = l^2$. Then if $(x_n) \in l_w^1(E)$, it follows that $\Sigma_n |\alpha_n| \, \|x_n\| < \infty$ for all $(\alpha_n) \in l^2$. Thus $(\|x_n\|) \in (l^2)^{\times} = l^2$.

Conversely, suppose $E$ has the Orlicz property. From the above discussion it is clear that each $(x_i) \in l_w^1(E)$ is unconditionally summable; thus $l_w^1(E) \subseteq l_s^2(E)$. Hence $\Sigma_{i=1}^{\infty} |\alpha_i| \, \|x_i\| < \infty$ for all $(x_i) \in l_w^1(E)$ and all $(\alpha_i) \in l^2$. Therefore we have $m(E) \supseteq l^2$. ∎

*Remark.* Since both $m(E)$ and $l^2$ are *BK*-spaces, it follows in particular that they are also topologically isomorphic if $E$ has the Orlicz property.

The Banach spaces which contain isomorphic copies of $c_0$ are excluded by the Orlicz property. But it is easily verified (using the fact that $(e_i) \in l_w^1(c_0)$) that $m(c_0) = l^1$. Thus it follows from Lemma 1.3 that $m(E) = l^1$ for all Banach spaces $E$ which contain isomorphic copies of $c_0$. Adjusting the proof of 2.3, we obtain conditions for $m(E) = l^q$ (with $1 \leq q < 2$) to hold:

PROPOSITION 2.4. *Let $1 \leq q < 2$. Suppose $E$ is an infinite dimensional Banach space such that*

(i) *there exists a real number $K > 0$ such that for all $(\alpha_i) \in l^p$ (with $\frac{1}{p} + \frac{1}{q} = 1$) there exists $(x_i) \in l_w^1(E)$ with $|\alpha_i| \leq K\|x_i\|$ for all $i$;*

(ii) $l_w^1(E) \subseteq l_s^p(E)$.

*Then $m(E) = l^q$. Being BK-spaces, it follows that the norms on $m(E)$ and $l^q$ are equivalent in this case.*

From a result of J. P. Kahane (cf. [3, p. 141]) it follows that property (ii) is satisfied by Banach spaces of cotype $p$. The space $l^p$ (with $2 \leq p < \infty$) is easily seen to satisfy both the properties (i) and (ii). So we may conclude that $m(l^p) = l^q$ where $\frac{1}{p} + \frac{1}{q} = 1$. Moreover, we have

PROPOSITION 2.5. *Let $E$ be an infinite dimensional $L^p$-space, where $1 \leq p \leq \infty$. Then $m(L^p) = l^s$ where $s = \min\{2, q\}$, with $\frac{1}{p} + \frac{1}{q} = 1$.*

*Proof.* When $1 \leq p \leq 2$, the space $E$ has cotype 2. Thus by Theorem 2.3 we have $m(E) = l^2$. For $2 < p < \infty$ the property (ii) in 2.4 is satisfied by Kahane's result, since $E$ has cotype $p$ in this case. This also follows from [5, Corollary 10.7, p. 200]. Furthermore, $l^p$ is topologically isomorphic to a closed subspace of $E$; thus there is an isomorphism $I: l^p \to E$ (into $E$) and a number $K > 0$ such that $\|(\alpha_i)\|_p \leq K\|I((\alpha_i))\|$ for all $(\alpha_i) \in l^p$. Let $(\alpha_i) \in l^p$. Put $x_i := I(\alpha_i e_i)$, for $i = 1, 2, \dots$. Then $|\alpha_i| \leq$

$K\|x_i\|$ for all $i$ and $(x_i) \in l_w^1(E)$ since $(\alpha_i e_i) \in l_w^1(l^p)$. Thus the property (i) in Proposition 2.4 holds. We have $m(L^p) = l^q$. When $p = \infty$, then $m(E) = l^1$, since then $E$ contains an isomorphic copy of $c_0$. ∎

From Lemma 1.3 follow some interesting observations, which we summarise in the following

PROPOSITION 2.6. *Let $E$ be a closed subspace of the Banach space $F$. Then*

(a) *If $E$ is complemented in $F$, then $m(F) \subseteq m(F/E)$.*

(b) *If $E$ is complemented in $F$, then $m(F^*) \subseteq m(E^*)$.*

(c) $m(F^*) \subseteq m(E^\perp) = m((F/E)^*)$.

For a fixed Banach space $X$, the sequence space $\lambda_X$ is defined as follows:

DEFINITION 2.7 (Marchena and Piñeiro [12]). A scalar sequence $(\alpha_i)$ belongs to $\lambda_X$ if and only if for every null sequence $(x_i)$ in $X$, the sequence $(\alpha_i x_i)$ lies in the range of some $X$-valued measure with bounded variation.

Following is one of the main results in [13]:

THEOREM 2.8 [13, p. 3329]. *Let $X$ be a Banach space. For a bounded sequence $(x_n)$ in $X$, consider the linear operator $T$: $(\alpha_n) \in l^1 \to \Sigma \alpha_n x_n \in X$. The following assertions hold:*

(i) *$(x_n)$ is in the range of an $X$-valued bv-measure if and only if $T$ is Pietsch-integral.*

(ii) *$(x_n)$ lies inside the range of a vector bv-measure (that is, there exist a Banach space $X_0$, an isometry $J$: $X \to X_0$, and a bv-vector measure $G$: $\Sigma \to X_0$ so that $J(x_n) \in \mathrm{rg}(G)$, $\forall n \in \mathbb{N}$) if and only if $T$ is 1-summing.*

(iii) *$(x_n)$ lies in the range of an $X^{**}$-valued bv-measure if and only if $T$ is integral.*

Applying Theorem 2.8, Marchena and Piñeiro proved (in [12]) the following characterisation of the sequence space $\lambda_X$:

THEOREM 2.9 [12, Theorem 1]. *Let $X$ be a Banach space and let $(\alpha_n)$ be a bounded scalar sequence. Then $(\alpha_n) \in \lambda_X$ if and only if $\Sigma_{i=1}^{\infty} |\alpha_i| \, \|x_i^*\|$ converges for all weakly unconditionally Cauchy series $\Sigma_n x_n^*$ in $X^*$.*

Clearly, the last result shows that

PROPOSITION 2.10. *For any Banach space $X$ we have $m(X^*) = \lambda_X$; i.e., a bounded scalar sequence $(\alpha_i)$ is in $m(X^*)$, if and only if for all null sequences $(x_i)$ in $X$ the sequence $(\alpha_i x_i)$ is in the range of some $X$-valued measure with bounded variation.*

Realising the relationship in 2.10, several results in connection with the space $\lambda_X$ (in [12]) now follow easily from corresponding results (with easy proofs) for the space $m(X^*)$, some of which are discussed in Aywa's thesis (cf. [1])—and of course, vice versa, some properties in connection with $m(X^*)$ can be obtained from [12]. As a matter of fact, a close look at the first part of the proof of 2.9 (as is discussed in [12]) reveals that if $(\alpha_i) \in \lambda_X$, then $\sum_{i=1}^{\infty} \alpha_i e_i^* \otimes x_i \in \mathscr{N}(l^1, X)$ for all $(x_i) \in (c_0)_s(X)$. Conversely, if $\sum_{i=1}^{\infty} \alpha_i e_i^* \otimes x_i \in \mathscr{N}(l^1, X) \subseteq \mathscr{P}\mathscr{I}(l^1, X)$ for all $(x_i) \in (c_0)_s(X)$, then by Theorem 2.8 the sequence $(\alpha_i x_i)$ is in the range of an $X$-valued *bv*-measure for all $(x_i) \in (c_0)_s(X)$—thus $(\alpha_i) \in \lambda_X$ in this case. Thus we have:

COROLLARY 2.11. *Let X be a Banach space. Then*

$$(\alpha_i) \in m(X^*) \qquad \Leftrightarrow \qquad \sum_{i=1}^{\infty} \alpha_i e_i^* \otimes x_i \in \mathscr{N}(l^1, X)$$

*for all* $(x_i) \in (c_0)_s(X)$.

In particular, this says that

COROLLARY 2.12. *Let X be a Banach space. Then*

$$(\alpha_i) \in m(X^{**}) \qquad \Leftrightarrow \qquad \sum_{i=1}^{\infty} \alpha_i e_i^* \otimes x_i^* \in \mathscr{N}(l^1, X^*)$$

*for all* $(x_i^*) \in (c_0)_s(X^*)$.

We know from Lemma 1.3 that $m(X^{**}) \subseteq m(X)$. The proof of the following lemma (which is the key to proving that indeed $m(X^{**}) = m(X)$ holds) is similar to the proof of Lemma 1.7.

LEMMA 2.13. *Let* $(\alpha_i) \in m(X)$. *Consider the bounded linear operator*

$$P\colon K(c_0, X) \to l_s^1(X)\colon \sum_{n=1}^{\infty} e_n^* \otimes x_n \mapsto (\alpha_n x_n).$$

$P^*$ *maps* $l_s^{\infty}(X^*)$ *into* $\mathscr{I}(X, c_0)$. *Moreover*, $P^*$ *maps* $(c_0)_s(X^*)$ *into* $\mathscr{N}(X, c_0)$ *and*

$$(P^*((x_i^*)))^* = \sum_{i=1}^{\infty} \alpha_i e_i^* \otimes x_i^*$$

*for all* $(x_i^*) \in (c_0)_s(X^*)$.

THEOREM 2.14. *Let X be a Banach space. Then*

$$m(X) = m(X^{**}).$$

*Proof.* We need only prove that $m(X) \subseteq m(X^{**})$. Let $(\alpha_i) \in m(X)$. It follows from Lemma 2.13 that

$$\sum_{i=1}^{\infty} \alpha_i e_i^* \otimes x_i^* \in \mathscr{N}(l^1, X^*)$$

for all $(x_i^*) \in (c_0)_s(X^*)$. Hence $(\alpha_i) \in m(X^{**})$ by Corollary 2.12. ∎

COROLLARY 2.15. *Let X be a Banach space. The following are equivalent*:

(i) $(\alpha_i) \in m(X)$

(ii) $\sum_{i=1}^{\infty} \alpha_i e_i^* \otimes x_i^* \in \mathscr{N}(l^1, X^*)$, *for all* $(x_i^*) \in (c_0)_s(X^*)$

(iii) $(\alpha_i) \in \lambda_{X^*}$

(iv) *For every null sequence* $(x_i^*)$ *in* $X^*$, *the sequence* $(\alpha_i x_i^*)$ *lies in the range of some* $X^*$*-valued measure with bounded variation.*

## 3. $(\Lambda, \Sigma)$-SUMMING MULTIPLIERS

Throughout this section we assume that the scalar sequence spaces $\Lambda$ and $\Sigma$ are normal symmetric *BK*-spaces with the *AK*-property.

DEFINITION 3.1. A scalar sequence $(\xi_i)$ is said to be a $(\Lambda, \Sigma)$-summing multiplier for a Banach space $E$ if $(\xi_i x_i) \in \Sigma_s(E)$ for all $(x_i) \in \Lambda_w(E)$.

Put

$$\begin{aligned} m_{\Lambda,\Sigma}(E) &= \{(\xi_i) \in w : (\xi_i x_i) \in \Sigma_s(E), \forall (x_i) \in \Lambda_w(E)\} \\ &= \{(\xi_i) \in w : (\|\xi_i x_i\|) \in \Sigma, \forall (x_i) \in \Lambda_w(E)\}. \end{aligned}$$

To see that $m_{\Lambda,\Sigma}(E) \subseteq l^\infty$, consider arbitrary $(\alpha_i) \in m_{\Lambda,\Sigma}(E)$ and let

$$T_n \colon \Lambda_w(E) \to \Sigma_s(E) :: (x_i) \mapsto (\alpha_i x_i)(\leq n).$$

Each $T_n$ has closed graph, and hence is a bounded linear operator. And

$$\pi_\Sigma\big((T_n((x_i)))\big) = \|(|\alpha_i| \, \|x_i\|)(\leq n)\|_\Sigma \leq \|(|\alpha_i| \, \|x_i\|)\|_\Sigma$$

for all $n$. The set $\{T_n : n = 1, 2, \dots\}$ is thus pointwise bounded, and hence also uniformly bounded. There exists $M > 0$ such that

$$\pi_\Sigma\big((T_n((x_i)))\big) \leq M \epsilon_\Lambda((x_i))$$

for all $n$. In particular, for any $x \in E$ such that $\|x\| = 1$, we have

$$|\alpha_i| = \|(0, \dots, 0, |\alpha_i| \, \|x\|, 0, 0, \dots)\|_\Sigma \leq M$$

for all $i = 1, 2, \ldots$ . Since the sequences in $\Lambda_w(E)$ are norm bounded in $E$, it is easy to see that $\Sigma \subseteq m_{\Lambda,\Sigma}(E)$.

On the vector space $m_{\Lambda,\Sigma}(E)$ we define a norm by

$$\begin{aligned}\|(\xi_i)\|_{\Lambda,\Sigma} &= \sup\{\pi_\Sigma((\xi_i x_i)) : \epsilon_\Lambda((x_i)) \leq 1\} \\ &= \sup\{\|(\|\xi_i x_i\|)\|_\Sigma : \epsilon_\Lambda((x_i)) \leq 1\}.\end{aligned}$$

A standard argument shows that

THEOREM 3.2. *$(m_{\Lambda,\Sigma}(E), \|\cdot\|_{\Lambda,\Sigma})$ is a complete normed space.*

The proof of the following generalisation of Theorem 1.4 will not be discussed in full detail (since it is similar to the proof of Theorem 1.4.), but for the sake of completeness we present an outline of the proof in this general context.

THEOREM 3.3. *Let $(\alpha_i)$ be a bounded scalar sequence. If $\Lambda$ is reflexive, then $(\alpha_i) \in m_{\Lambda,\Lambda}(E^*)$ if and only if the operator $\Lambda \to E$: $(\beta_i) \to \sum_{i=1}^{\infty} \beta_i \alpha_i x_i$ is integral for all sequences $(x_i) \in \Lambda_s^\times(E)$.*

*Proof.* Let $(\alpha_i) \in m_{\Lambda,\Lambda}(E^*)$. As before, let

$$P: K(E, \Lambda) \to \Lambda_s(E^*) :: P\left(\sum_{n=1}^{\infty} x_n^* \otimes e_n\right) = (\alpha_n x_n^*),$$

now using the isometry $K(E, \Lambda) \cong \Lambda_c(E^*)$ (cf. [8]). Then $P$ is linear and bounded and

$$\begin{aligned}P^*: \Lambda_s^\times(E^{**}) &\to K(E, \Lambda)^* = \mathscr{I}(\Lambda, E^{**}) :: \\ ((x_n)) &\mapsto \sum_n e_n^* \otimes \alpha_n x_n \in \mathscr{I}(\Lambda, E^{**})\end{aligned}$$

for all $(x_n) \in \Lambda_s^\times(E)$. Since $(\sum_n e_n^* \otimes \alpha_n x_n)(\Lambda) \subseteq E$ for all $(x_n) \in \Lambda_s^\times(E)$, we have

$$\sum_n e_n^* \otimes \alpha_n x_n \in \mathscr{I}(\Lambda, E), \qquad \forall (x_n) \in \Lambda_s^\times(E).$$

Since $\Lambda^\times$ has $AK$, it follows conversely that if $\Lambda \to E$: $(\beta_i) \mapsto \sum_i \beta_i \alpha_i x_i$ is integral for all $(x_i) \in \Lambda_s^\times(E)$ and

$$Q: \Lambda_s^\times(E) \to \mathscr{I}(\Lambda, E) :: Q((x_n)) = \sum_n e_n^* \otimes \alpha_n x_n \in \mathscr{I}(\Lambda, E),$$

then $Q$ is continuous into the isometric subspace $\mathscr{N}(\Lambda, E)$. Thus

$$Q^*: L(E, \Lambda) \to \Lambda_s(E^*)$$

is bounded. Using the trace duality and the fact that $\Lambda_w(E^*) \cong L(E, \Lambda)$ (cf. [8]), we have

$$(\alpha_n x_n^*) = Q^*((x_n^*)) \in \Lambda_s(E^*)$$

for all $(x_n^*) \in \Lambda_w(E^*)$. ▮

In general the *BK*-space $m_{\Lambda,\Sigma}(E)$ may not have the *AK*-property. The last part of this section is devoted to an attempt to find some conditions on $E$ and the relevant sequence spaces that will ensure that $m_{\Lambda,\Sigma}(E)$ has *AK*. Let us denote the unit balls in $\Lambda_c(E)$ and $(\Sigma_s(E))^* = \Sigma_s^{\times}(E^*)$ by $B_\Lambda^c$ and $B_{\Sigma^\times}^*$, respectively. The unit ball in $m_{\Lambda,\Sigma}(E)$ will be denoted by $B_{\Lambda,\Sigma}$.

Using the *AK*-property of the sequence space $\Sigma$, it is easy to see that

LEMMA 3.4. *For all* $(\xi_i) \in m_{\Lambda,\Sigma}(E)$, *we have*

$$\|(\xi_i)\|_{\Lambda,\Sigma} = \sup\{\pi_\Sigma((\xi_i x_i)) : (x_i) \in \Lambda_c(E), \epsilon_\Lambda((x_i)) \leq 1\}.$$

THEOREM 3.5. $m_{\Lambda,\Sigma}(E)$ *has the AK-property if and only if the set*

$$A := \{(\langle x_i, a_i\rangle) : ((x_i), (a_i)) \in B_\Lambda^c \times B_{\Sigma^\times}^*\}$$

*is* $\sigma(m_{\Lambda,\Sigma}(E)^\times, m_{\Lambda,\Sigma}(E))$ *relatively compact.*

*Proof.* First we show that $A \subseteq m_{\Lambda,\Sigma}(E)^\times$. Let $(\alpha_i) \in m_{\Lambda,\Sigma}(E)$ and $(\langle x_i, a_i\rangle) \in A$. Then, using the sign-function and the fact that $\Sigma$ is normal and symmetric, it follows that

$$\sum_{i=1}^{\infty} |\alpha_i \langle x_i, a_i\rangle| \leq \pi_\Sigma((\alpha_i x_i))\pi_{\Sigma^\times}((a_i)) < \infty.$$

Since $(\alpha_i) \in m_{\Lambda,\Sigma}(E)$ was arbitrarily chosen, it follows that $A \subseteq m_{\Lambda,\Sigma}(E)^\times$.

Next we show that $A^\circ = B_{\Lambda,\Sigma}$. Let $(\alpha_i) \in A^\circ$. For any $(x_i) \in B_\Lambda^c$ we have

$$\pi_\Sigma((\alpha_i x_i)) = \sup_{\pi_{\Sigma^\times}((a_i)) \leq 1} |\langle(\alpha_i x_i), (a_i)\rangle| = \sup_{\pi_{\Sigma^\times}((a_i)) \leq 1} \left|\sum_{i=1}^{\infty} \alpha_i \langle x_i, a_i\rangle\right| \leq 1.$$

Thus $A^\circ \subseteq B_{\Lambda,\Sigma}$.

Conversely, if $(\alpha_i) \in B_{\Lambda,\Sigma}$ and $(\langle x_i, a_i\rangle) \in A$, then

$$\left|\sum_{i=1}^{\infty} \alpha_i \langle x_i, a_i\rangle\right| = |\langle(\alpha_i x_i), (a_i)\rangle| \leq \pi_\Sigma((\alpha_i x_i))\pi_{\Sigma^\times}((a_i)) \leq 1,$$

thereby establishing the inclusion $B_{\Lambda,\Sigma} \subseteq A^\circ$.

Assume that $A$ is *weak** compact (in the Köthe duality). Since $A^\circ$ is a $\delta(m_{\Lambda,\Sigma}(E), m_{\Lambda,\Sigma}(E)^\times)$ neighbourhood of the origin (where $\delta$ denotes the topology of uniform convergence on the weak* compact sets), the equality $A^\circ = B_{\Lambda,\Sigma}$ implies that the norm topology is weaker than the $\delta$-topology on $m_{\Lambda,\Sigma}(E)$. By a result of G. Bennett (cf. [11, Theorem 2.1, p. 188]) the normal sequence space $m_{\Lambda,\Sigma}(E)$ has $AK$ with respect to this $\delta$-topology. Hence $m_{\Lambda,\Sigma}(E)$ has $AK$ with respect to its (weaker) norm topology. In particular, this shows that $m_{\Lambda,\Sigma}(E)^* = m_{\Lambda,\Sigma}(E)^\times$.

Conversely, let $m_{\Lambda,\Sigma}(E)$ have the $AK$-property. Then $m_{\Lambda,\Sigma}(E)^* = m_{\Lambda,\Sigma}(E)^\times$. Since $A^\circ = B_{\Lambda,\Sigma}$, we have $A \subseteq B^\circ_{\Lambda,\Sigma}$, which implies that $A$ is equicontinuous. Thus $A$ is $\sigma(m_{\Lambda,\Sigma}(E)^\times, m_{\Lambda,\Sigma}(E))$ relatively compact. ∎

Consider the bilinear mapping

$$\Phi\colon B^c_\Lambda \times B^*_{\Sigma^\times} \to m_{\Lambda,\Sigma}(E)^\times :: ((x_i),(a_i)) \mapsto (\langle x_i, a_i\rangle).$$

On $B^c_\Lambda$ and $B^*_{\Sigma^\times}$ consider the restrictions of the $\sigma(\Lambda_c(E), \Lambda_c(E)^*)$ and $\sigma(\Sigma_s^\times(E^*), \Sigma_s(E))$ topologies, respectively, and on $m_{\Lambda,\Sigma}(E)^\times$ consider the $\sigma(m_{\Lambda,\Sigma}(E)^\times, m_{\Lambda,\Sigma}(E))$ topology. We show that $\Phi$ is separately continuous. Let $(x_i^\delta)_i \to (x_i)_i$ in $B^c_\Lambda$ weakly. Fix $(a_i) \in B^*_{\Sigma^\times}$ and $(\lambda_i) \in m_{\Lambda,\Sigma}(E)$. Because of

$$\left|\sum_{i=1}^{\infty} \langle y_i, \lambda_i a_i\rangle\right| \le \sum_{i=1}^{\infty} |\langle y_i, \lambda_i a_i\rangle| = \sum_{i=1}^{\infty} |\langle \lambda_i y_i, a_i\rangle| < \infty$$

for all $(y_i) \in \Lambda_w(E)$, it follows from a result in [9] that $(\lambda_i a_i) \in \Lambda_c(E)^*$. Therefore

$$\sum_i \lambda_i \langle x_i^\delta, a_i\rangle = \langle (x_i^\delta), (\lambda_i a_i)\rangle \overset{\delta}{\to} \langle (x_i), (\lambda_i a_i)\rangle = \sum_i \lambda_i \langle x_i, a_i\rangle.$$

Since this holds for all $(\lambda_i) \in m_{\Lambda,\Sigma}(E)$, we have that

$$\Phi\big((x_i^\delta),(a_i)\big) = \big(\langle x_i^\delta, a_i\rangle\big) \overset{\delta}{\to} \big(\langle x_i, a_i\rangle\big) = \Phi\big((x_i),(a_i)\big)$$

in $m_{\Lambda,\Sigma}(E)^\times$ with the weak*-topology. Therefore $\Phi$ is continuous in the first component.

Similarly, let $(a_i^\delta)_i \to (a_i)_i$ in $B^*_{\Sigma^\times}$. For all $(\lambda_i) \in m_{\Lambda,\Sigma}(E)$ we have

$$\begin{aligned}\sum_i \lambda_i \langle x_i, a_i^\delta\rangle = \langle (\lambda_i x_i), (a_i^\delta)\rangle &\overset{\delta}{\to} \langle (\lambda_i x_i), (a_i)\rangle \\ &= \sum_i \langle \lambda_i x_i, a_i\rangle.\end{aligned}$$

Thus it follows that

$$\Phi\big((x_i),(a_i^\delta)\big) = \big(\langle x_i, a_i^\delta\rangle\big) \overset{\delta}{\to} \big(\langle x_i, a_i\rangle\big) = \Phi\big((x_i),(a_i)\big)$$

in the weak*-topology. Therefore $\Phi$ is continuous in the second component.

The separately continuous $\Phi$ maps compact sets of the form $K_1 \times K_2$ with both $K_1$ and $K_2$ compact, onto compact sets. If $\Lambda_c(E)$ is reflexive, then the unit ball $B_\Lambda^c$ is weakly compact. The set $B_{\Sigma^\times}^*$ is weak* compact. So $A = \Phi(B_\Lambda^c \times B_{\Sigma^\times}^*)$ is weak* compact. Thus from Theorem 3.5 it is clear that

THEOREM 3.6. *$m_{\Lambda,\Sigma}(E)$ has AK if one of the following holds*:

(a) *$\Lambda_c(E)$ is reflexive.*

(b) *The dual pair $(m_{\Lambda,\Sigma}(E), m_{\Lambda,\Sigma}(E)^\times)$ is barrelled.*

If both $\Lambda$ and $E$ are reflexive Banach spaces, then we know that $\Lambda_w^\times(E) = L(\Lambda, E)$ is reflexive if and only if $L(\Lambda, E) = K(\Lambda, E)$. Let $B_{\Lambda^\times}^c$ denote the unit ball in $\Lambda_c^\times(E)$. It follows that:

LEMMA 3.7. *Let $\Lambda$ be a reflexive BK space with AK. The set $B_{\Lambda^\times}^c$ is weakly compact $\Leftrightarrow$ $\Lambda_c^\times(E)$ is reflexive $\Leftrightarrow \Lambda_c^\times(E) = \Lambda_w^\times(E)$ and E is reflexive.*

Thus we conclude that:

THEOREM 3.8. *If E and the BK space $\Lambda$ (with AK) are both reflexive Banach spaces such that $L(\Lambda, E) = K(\Lambda, E)$, then $m_{\Lambda^\times,\Sigma}(E)$ has the AK-property.*

For $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$, it follows from a result in [2] that

$$L(l^p, E) = K(l^p, E) \quad \Leftrightarrow \quad l_w^q(E) = l_c^q(E)$$
$$\Leftrightarrow \quad l_w^q(E) \subset (c_0)_s(E).$$

COROLLARY 3.9. *Let $1 < p < \infty$ and suppose $\frac{1}{p} + \frac{1}{q} = 1$. If E is reflexive and $l_w^q(E) \subset (c_0)_s(E)$, then $m_{p,r}(E)$ has the AK-property for all $1 \le r < \infty$.*

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