(Knowledge for Development)

KIBABII UNIVERSITY

UNIVERSITY EXAMINATIONS

2020/2021 ACADEMIC YEAR

FOURTH YEAR FIRST SEMESTER

SPECIAL/SUPPLIMENTARY EXAMINATION

FOR THE DEGREE OF BACHELOR OF SCIENCE

COURSE CODE: MAT 407

COURSE TITLE: FUNCTIONAL ANALYSIS

DATE: 7/01/2022 **TIME:** 8:00 - 10:00 AM

INSTRUCTIONS TO CANDIDATES

Answer Question One and Any other TWO Questions

TIME: 2 Hours

QUESTION ONE (30marks)

a). Define the following terms

i). Relatively open set (2 Marks)

ii). An closed ball in a metric space (2 Marks)

iii). A uniformly bounded set (2 Marks)

iv). Aa normed linear space (4 Marks)

b). Prove that the $\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$ is a norm (7 Marks)

c). Show that the sequence $\{x_n\}$ where $x_n = \frac{1}{n}$ is Cauchy in $(0,1)$ but $(0,1)$ is not complete. (5 Marks)

d). Let $B(X)$ be the set of bounded functions on a set X. Prove that the space $(B(X), \|\cdot\|_\infty)$ is complete. (8 Marks)

QUESTION TWO (20marks)

a). Let X be any non empty set and $\{f_n\}$ a sequence of bounded functions on X such that $|f_n(x)| \leq M_n > 0$ for all $n \geq 1$ and all $x \in X$. Prove that $\sum_{n=1}^{\infty} f_n$ is uniformly convergent if $\sum_{n=1}^{\infty} M_n < \infty$. (9 Marks)

b). Discuss the results of Arzela-Ascoli theorem with respect to compactness in continuous functions. Does the results relates to the Heine –Borel theorem on $\mathbb{R}$. (6 Marks)

c). Prove that $\mathbb{N}$ is nowhere dense in $\mathbb{R}$. (5 Marks)

QUESTION THREE (20marks)

a). State and prove the Cantor intersection theorem (8 Marks)

b). Consider the set $A = (-1,1)$.

(i). Identify any compact set B such that $B \subset A$ (2 Marks)

(ii). Show that B is bounded. Explain (4 Marks)

(iii). Is B uniformly bounded? Explain (3 Marks)

(iv). Show that B is dense in $\mathbb{R}$ (3 Marks)

QUESTION FOUR (20marks)

a). (i). What do you understand by the term isometric metric spaces? (2 Marks)

(ii). Show that the map from $\mathbb{C}$ to $\mathbb{R}^2$ given by $z = x + iy \mapsto (x, y)$ for $i = \sqrt{-1}$ is isometric.

(5 Marks)

b). Let V be an inner product space. Prove that $|\langle x, y\rangle| \leq \|x\|\|y\|$ for all $x, y \in V$. (6 Marks)

c). Prove that any continuous function from a compact metric space to any other metric space is uniformly continuous. (7 Marks)

QUESTION FIVE (20marks)

a). Define a compact set hence show that $\mathbb{R}$ is not compact. (4 Marks)

b). (i). When do we say a collection of sets have a finite intersection property. (2 Marks)

(ii). Suppose X is a metric space and $\mathcal{C}$ is a non-empty collection of compact subsets of X. If $\mathcal{C}$ has finite intersection property, then $\bigcap \mathcal{C} \neq \phi$. (4 Marks)

c). Suppose that X and Y are metric spaces where X is compact and $f: X \to Y$ is continuous. Show that $f(X)$ is bounded. (6 Marks)

d). Define a totally bounded set hence prove that $A = (0,1)$ is totally bounded. (4 Marks)