(Knowledge for Development)

KIBABII UNIVERSITY

UNIVERSITY EXAMINATIONS

2020/2021 ACADEMIC YEAR

THIRD YEAR FIRST SEMESTER

SPECIAL/SUPPLIMENTARY EXAMINATION

FOR THE DEGREE OF BACHELOR OF EDUCATION AND BACHELOR OF SCIENCE

COURSE CODE: **MAP312/MAT 303**

COURSE TITLE: LINEAR ALGEBRA III

DATE: 14/01/2022 **TIME:** 11:00 AM - 1:00 PM

INSTRUCTIONS TO CANDIDATES

Answer Question One and Any other TWO Questions

TIME: 2 Hours

QUESTION ONE (30 MKS)

(a). Define the following terms

(i). Diagonalizable matrix (1 mk)

(ii). Unitary matrix (1 mk)

(iii). Bilinear form (3 mks)

(b). Prove that $\begin{bmatrix} 2 & i-1 & 2i \\ -1-i & 1 & i \\ -2i & -i & -3 \end{bmatrix}$ is a hermitian matrix. (2 mks)

(c). (i). Define the complex vectors $u, v \in \mathbb{C}^3$ as

$$u = \langle 2+i, 0, 4-5i \rangle, \;\; v = \langle 1+i, 2+i, 0 \rangle.$$

Determine the Euclidean norms $\|u\|$ *and* $\|v\|$. (4 mks)

(ii). Let $\mathbb{C}^n$ be a complex vector space where $u, v \in \mathbb{C}^n$. If $\bar{u}$ and $\bar{v}$ denotes the conjugates of u and v respectively, show that $\overline{u-v} = \bar{u} - \bar{v}$. (3 mks)

(d). Let $T: V \to V$ be an operator whose characteristic polynomial $\Delta(t) = (t-4)^6$ and minimum polynomial $m(t) = (t-4)^3$. Determine all possible Jordan Canonical

forms for T. (4 mks)

(e). Determine the matrices P and D such that $D = P^{-1}AP$ where $A = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$. (6 mks)

(f).(i). What do you understand by the term, a quadratic form? (2 mks)

(ii). Determine the definiteness of the quadratic form $Q(x_1, x_2, x_3) = 3x_1^2 + 2x_2^2 + 3x_3^2 - 2x_1x_2 - 2x_2x_3$. (4 mks).

QUESTION TWO (20 MKS)

(a). Differentiate between algebraic and geometric multiplicity of an eigenvalue of a

matrix A. (2 mks)

(b). (i). What is symmetric matrix? (1 mks) (ii). Orthogonally diagonalize matrix $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ (17 mks)

QUESTION THREE (20 MKS)

(a). Let A and B be an $n \times m$ and $m \times p$ complex matrices respectively. If $\bar{A}$ and $\bar{B}$ are complex conjugates of A and B respectively, prove that $\bar{\bar{A}} = \mathrm{A}$ and $\overline{AB} = \bar{A}\bar{B}$. (5 mks).

(b). Prove that eigenvectors of a real symmetric matrix are orthogonal. (3 mks)

(c). Show that $A = \frac{1}{2}\begin{bmatrix} 1 & -i & -1+i \\ i & 1 & 1+i \\ 1+i & -1+i & 0 \end{bmatrix}$ is a unitary matrix hence find its inverse.

(5 mks)

(d). (i). Define an orthonormal set of vectors in $\mathbb{R}^n$. (2 mks)

(ii). Prove that if A is an $n \times n$ orthogonal matrix, then the row as well as the column vectors of A forms an orthonormal set in $\mathbb{R}^n$ with the Euclidean inner product. (5 mks)

QUESTION FOUR (20 MKS)

(a). (i). Define a nilpotent matrix hence show that $N = \begin{bmatrix} 0 & 2 & 1 & 6 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (3 mks)

(ii). Show that a matrix A is nilpotent if and only if all its eigenvalues are zero. (6 mks)

(b). Find the orthogonal change of variable that eliminates the cross product term in the quadratic form $Q(x_1, x_2) = 8x_1^2 - 4x_1x_2 - 5x_2^2$ and expresses it in terms of new variables.

(6mks)

(c). Let U be unitary matrix. Prove that $\|Ux\| = \|x\|$ hence $\langle Ux, Uy\rangle = \langle x, y\rangle$ for $x, y \in \mathbb{R}^n$.

(5 mks)

QUESTION FIVE (20 MKS)

(a).(i). Differentiate between orthogonal and Hermitian matrix. (2 mks)

(ii). Prove that an orthogonal matrix is Isometric. (4 mks)

(b). Let λ be an eigenvalue of a real $n \times n$ matrix B, and x the corresponding eigenvector. Show that if $\bar{\lambda}$ is also an eigenvalue of B and $\bar{x}$ is a corresponding eigenvector. (3 mks) (c). Let P and Q be linear transformations on complex vector space V such that $P: V \to V$ and $: V \to V$. Prove that $(P+Q)^* = P^* + Q^*$and $(PQ)^* = P^*Q^*$ (5 mks)

(d). Prove that the real matrix $Q = \begin{bmatrix} p & -q \\ q & p \end{bmatrix}$ has the eigenvalues $\lambda = p \pm qi$ and if p and q are not all zeros, then $\begin{bmatrix} p & -q \\ q & p \end{bmatrix} = \begin{bmatrix} |\lambda| & 0 \\ 0 & |\lambda| \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ where θ is argument of λ. (6 mks)