(Knowledge for Development)

KIBABII UNIVERSITY

UNIVERSITY EXAMINATIONS

2020/2021 ACADEMIC YEAR

THIRD YEAR FIRST SEMESTER

SPECIAL/SUPPLEMENTARY EXAMINATION

FOR THE DEGREE OF BACHELOR OF SCIENCE IN MATHEMATICS

COURSE CODE: **MAP 313/MAP324**

COURSE TITLE: GROUP THEORY

DATE: 11/01/22 **TIME:** 8 AM -10 AM

INSTRUCTIONS TO CANDIDATES

Answer Question ONE and Any other TWO Questions

TIME: 2 Hours

This Paper Consists of 4 Printed Pages. Please Turn Over.

QUESTION ONE (30 MARKS)

a) Define the following

  i. Abelian group (2 marks)

  ii. Subgroup (2 marks)

  iii. Group homomorphism (2 marks)

  iv. Isomorphic groups (2 marks)

b) Show that $Ha = Hb$ if and only if $ab^{-1} \in H$ for $a,b \in G$. Similarly, $aH = bH$ if and only if $a^{-1}b \in H$ for $a,b \in G$. (6 marks)

c) Show that if H is a subgroup of G, then $|G| = |H|[G : H]$. In particular, if G is finite then $|H|$ divides $|G|$ and $[G : H] = |G|/|H|$. (5 marks)

d) Let G be a finite group. Show that if $a \in G$, then $|a|$ divides $|G|$.

In particular, $a^{|G|} = 1$. (3marks)

e) Let G be a finite group. Show that If G has prime order, then G is cyclic. (3marks)

f) Show that if H is normal in G, then the cosets of H form a group. (5marks)

QUESTION TWO (20 MARKS)

a) Define the following

  i. *Quotient group* (2 marks)

  ii. The index of a subgroup (2marks)

  iii. Normal subgroup (2 marks)

b) Let G be a group. Show that Every normal subgroup of G is the kernel of a homomorphism

(5 marks)

c) Let f: G → H be a homomorphism. Show that if K is a subgroup of G, then f(K) is a subgroup of H. If f is an epimorphism and K is normal, then f(K) is normal. (5 marks)

d) Let f: G → H be a homomorphism. Show that If K is a subgroup of H, then $f^{-1}(K) = \{x \in G, f(x) \in K\}$ is a subgroup of G. If K is normal, so is $f^{-1}(K)$. (4marks)

QUESTION THREE (20 MARKS)

a) Define the following

   i. Symmetric group (2 marks)

   ii. Alternating group (2marks)

b) Show that every group is isomorphic to a group of permutations. (8 marks)

c) Suppose that a group G acts on a set X. Let B(x) be the orbit of x ∈ X, and let Stab(x) be the stabilizer of x. Then the size of the orbit is the index of the stabilizer that is $|B(x)| = [G: Stab(x)]$. If G is finite, then $|B(x)| = |G|/|Stab(x)|$. In particular, the size of an orbit divides the order of the group. (8 marks)

QUESTION FOUR (20MARKS)

a) Define the following

   i. The order of an element (2 marks)

   ii. Cyclic group (2marks)

   iii. Coset (2 marks)

b) Let the finite group G act on the finite set X, and denote by X^g the set of elements of X that are fixed by g, that is $X^g = \{x \in X, g.x = x\}$. Show that the number of orbits $= \frac{1}{|G|}\sum_{g\in G}|x^g|$. (6marks)

c) Let G be a finite group of order n such that all its non-trivial elements have order 2. Show that G is abelian. (4 marks)

d) Let H be a subgroup of G, and let $g \in G$ but not in H. Show that $H \cup gH$ is a subgroup of G. (4 marks)

QUESTION FIVE (20 MARKS)

a) Define the following

i. an orbit (2marks)

ii. transitive (2marks)

iii. stabilizer (2marks)

b) Let $G = S_3$ be the group of permutations of 3 elements, that is G = {(1), (12), (13), (23), (123), (132)} and let H = {(1), (12)} be a subgroup. Compute the left and right cosets of H. (6 marks)

c) Let G be a finite group and let H and K be subgroups with relatively prime order. Show that $H \cap K = \{1\}$. (4marks)

d) Let G be a group. Show that if H is a normal subgroup of order 2, then H belongs to the center of G. (4 marks)