(*Knowledge for Development*)

KIBABII UNIVERSITY

UNIVERSITY EXAMINATIONS

2021/2022 ACADEMIC YEAR

FIRST YEAR FIRST SEMESTER

SPECIAL EXAMINATION

FOR THE DEGREE OF MASTER OF SCIENCE IN MATHEMATICS

COURSE CODE: **MAT 824**

COURSE TITLE: OPERATOR THEORY I

DATE: 27/07/2022 **TIME:** 8:00 AM -10:00 AM

INSTRUCTIONS TO CANDIDATES

Answer Question ONE and Any TWO Questions

TIME: 2 Hours

This Paper Consists of **3 Printed Pages**. Please Turn Over.

QUESTION ONE (20 MARKS)

a. Define the following
   i. Inner product (3 mark)
   ii. Norm (3 mark)

b. Show that if $\langle . \rangle$ is a an inner product on the complex vector space V then $|\langle x, y\rangle|^2 \leq \langle x, x\rangle\langle y, y\rangle$ for every $x, y \in V$ (6 marks)

c. Show that if $\langle . \rangle$ is an inner product on the complex vector space V then $\|x\| = \langle x, x\rangle^{\frac{1}{2}}$ defines a norm on V. (8marks)

QUESTION TWO (20 MARKS)

a. Define the following
   i. Open and closed sets (3 mark)
   ii. Convergence (2 mark)
   iii. Compactness (2marks)

b. Let $\langle . \rangle$ be an inner product on the complex vector space V, with corresponding norm $\|.\|$. Show that $\langle x, y\rangle = \frac{1}{4}\sum_{n=0}^{3} i^{-n} \|x + i^n y\|^2$ for every $x, y \in V$ (4 marks)

c. Show that if T ∈ B (H) is such that $\langle x, Tx\rangle$= 0 for all x ∈ H then T = 0. (5 marks)

QUESTION THREE (20 MARKS)

a. Define the following
   i. Cauchy sequence (3 mark)
   ii. Bounded linear transformation (2marks)

b. Let Y be a subspace of the Banach space X. Show that Y is closed if and only if Y is complete. (5marks)

c. Show that A linear transformation T ∈ L (H; K) is continuous if and only if there exists M > 0 such that $\|Tx\| \leq M\|x\|$ for every x ∈ H. (6marks)

d. Show that the set B(H; K) is a subspace of L(H; K) (4marks)

QUESTION FOUR (20 MARKS)

a. Define the following
   i. Kernel and range (4 mark)
   ii. Orthogonal projection (2marks)

b. Show that if T ∈ B (H; K) then ker T is a closed subspace of H. (3marks)

c. If D ⊆ H then $D^{\perp}$ is a closed subspace of H. (5marks)

d. Show that If L ⊆ H is a closed subspace of H then $L = (L^{\perp})^{\perp}$. (6marks)

QUESTION FIVE (20 MARKS)

a. Define the following
   i. Invertible operator (2 mark)
   ii. Spectrum of an operator (2marks)

b. Show that the orthogonal projection $P_L \in B(H)$ and is such that $P_L^2 = P_L = P_L^*$. (11 marks)

c. Let $T \in B(H; K)$. Show that there exists at most one operator $S \in B(K; H)$ such that $ST = I$ and $TS = I$. (3marks)

d. Let $S \in B(H; K)^\times$ and $T \in B(K; L)^\times$. Show that the operator $TS \in B(H; L)^\times$, with $(TS)^{-1} = S^{-1}T^{-1}$. (2marks)