(Knowledge for Development)

KIBABII UNIVERSITY

UNIVERSITY EXAMINATIONS

2021/2022 ACADEMIC YEAR

FIRST YEAR SECOND SEMESTER

SUPPLEMENTARY EXAMINATION

FOR THE DEGREE OF MASTER OF SCIENCE IN PURE MATHEMATICS

COURSE CODE: MAT 813

COURSE TITLE: FUNCTIONAL ANALYSIS I

DATE: 22/07/2022 **TIME:** 8:00 AM - 10:00 AM

INSTRUCTIONS TO CANDIDATES

Answer Any other THREE Questions

TIME: 2 Hours

QUESTION ONE (20 MARKS)

a) Define the following

   i. Metric space

   ii. Conjugate component

   iii. Open ball

   iv. closed ball

   v. sphere

b) Show that $\mathcal{L}^p$ is a Metric space

c) Show that a mapping T of a metric space X into a metric space Y is continuous if and only if the inverse image of any open subset of Y is an open subset of X

QUESTION TWO (20 MARKS)

a) Define the following terms

   i. Convergent sequence

   ii. Bounded sequence

b) Show that every convergent sequence in a metric space is a Cauchy sequence

c) Show that the dual space of l^l is l^∞

d) State the open mapping theorem

QUESTION THREE (20 MARKS)

a) Given f is a bounded linear functional on a subspace Z of a normed space X, show that there exists a bounded linear functional $\bar{f}$ on X which is an extension of f to X and has the same norm $||\bar{f}||_x = ||f||_z$ where $||\bar{f}||_x = \sup|\bar{f}(x)|_{x\in X}, \left||x|\right| = 1$ $\left||\bar{f}|\right|_z = \sup|\bar{f}(x)|_{x\in Z}, \left||x|\right| = 1$ and $\left||f|\right|_z = 0$ in the trivial case $z = \{0\}$

b) Given X is a normed space and $x_0 \neq 0$ is any element of X, show that there exists a bounded linear function $\bar{f}$ on X such that $\left||f|\right| = 1$ and $\bar{f}(x_0) = ||x_0||$

QUESTION FOUR (20 MARKS)

a) Given (T_n) is a sequence of bounded linear operators $T_n: X \to Y$ from a Banach space X into a normed space Y such that $(||T_n x||)$ is bounded for every $x \in X$ say $\left||T_n x|\right| \leq c_n$ where c_n is a real number. Show that the operator of the norm $\left||T_n|\right|$ is bounded. That is, there is a c such that $\left||T_n|\right| \leq c, \; n = 1,2,3, \ldots$

b) Given (x_n) is a weakly convergent sequence in a normed space X, say $x_n \xrightarrow{w} x$ show

i. The weak limit x of (x_n) converges weakly to x

ii. The sequence $||(x_n)||$ is bounded

QUESTION FIVE (20 MARKS)

a) Define the following terms

i. Uniformly operator convergent

ii. Strongly operator convergent

iii. Weakly operator convergent

b) Show that an A-summability method with matrix $A = (\propto_{nk})$ is regular if and only if

i. $\lim_{n\to\infty} \propto_{nk} = 0$ for $k = 1,2\ldots.$

ii. $\lim_{n\to\infty} \sum_{k=1}^{\infty} \propto_{nk} = 0$

iii. $\sum_{k=1}^{\infty} | \propto_{nk}| \leq r$ for n=1,2 where r is a constant which does not depend on n