(Knowledge for Development)

KIBABII UNIVERSITY

UNIVERSITY EXAMINATIONS

2020/2021 ACADEMIC YEAR

FOURTH YEAR FIRST SEMESTER

SPECIAL/SUPPLIMENTARY EXAMINATION

FOR THE DEGREE OF BACHELOR OF SCIENCE

COURSE CODE: **MAT 405**

COURSE TITLE: MEASURE THEORY

DATE: 13/01/2022 **TIME:** 8:00 - 10:00 AM

INSTRUCTIONS TO CANDIDATES

Answer Question One and Any other TWO Questions

TIME: 2 Hours

This Paper Consists of 4 Printed Pages. Please Turn Over.

Question 1 (30 marks) – Compulsory

a) If the sets E_1 *and* E_2 are measurable, then their union is also measurable. Prove (7 mks)

b) Show that the interval (a, ∞) is measurable. (10 mks)

c) If $M^* E = 0$, then the set E is measurable. Prove (7 mks)

d) $f(x) = \begin{cases} \frac{1}{x} & 0 < x \leq 1 \\ 9 & x = 2 \end{cases}$

Show that the function f is not lebesque integrable. (6 mks)

Question 2 (20 marks)

a) Show that if A_1 *and* A_2 are measurable subsets of $[a, b]$, then $A_1 - A_2$ is measurable and if $A_2 \subseteq A_1$ show that $M(A_1 - A_2) = MA_1 - MA_2$. (5 mks)

b) A necessary and sufficient condition for a set A to be measurable is that for all $\varepsilon > 0$, there exists an open set F containing A and a closed set B contained in A such that $MF - MB < \epsilon$. Prove (8 mks)

c) $f(x) = \begin{cases} 1 & x \text{ is rational} \\ 0 & x \text{ is irrational} \end{cases}$

Show that this function is Lebesque integrable but is not Riemann integrable. (7 mks)

Question 3 (20 marks)

a) A necessary and sufficient condition for a bounded function f to be Lebesque integrable over the interval $[a, b]$ is that for each given $\varepsilon > 0$, there exists a measurable partition P of the interval $[a, b]$ such that $U(P, f) - L(P, f) < \varepsilon$. Prove. (10 mks)

b) Every bounded measurable function in the interval $[a, b]$ is Lebesque integrable on that interval. Prove. (10 mks)

Question 4 (20 marks)

a) Define the term Lebesque integral. (3 mks)

b) Let f be a bounded function on the interval $[a, b]$, then for any two measurable partitions of the interval $[a, b]$, we have $(P_{1,}f) \geq L(P_{2,}f)$; $L \int_{-a}^{b} f dx < L \int_{a}^{-b} f dx$. Prove (6 mks)

c) If the set F is measurable, then the absolute value of F is also measurable. (4 mks)

d) If f is measurable on the interval $[a, b]$ and if K is real, then $f(x) + K$ and $Kf(x)$ are also measurable. (7 mks)

Question 5 (20 marks)

a) Show that every bounded Riemann integrable function over the interval $[a, b]$ is Lebesque integrable and the two integrals are the same. (6 mks)
b) If the function $f = g\ a.e$ and f is measurable then g is measurable. (5 mks)
c) Show that if the function $f(x)$ is measurable, then the set $\{x: f(x) = \alpha\}$, $\alpha \epsilon \mathbb{R}$is measurable for each extended real number α. (5 mks)
d) Show that every continuous function is measurable. (4 mks)