KIBABII UNIVERSITY

UNIVERSITY EXAMINATIONS

2021/2022 ACADEMIC YEAR

SECOND YEAR SECOND SEMESTER FOR THE DEGREE OF

BACHELOR OF EDUCATION SCIENCE

COURSE CODE: **SPH 222**

COURSE TITLE: **QUANTUM MECHANICS**

DATE: 10/05/2022 **TIME: 9:00AM-11:00AM**

INSTRUCTIONS TO CANDIDATES

Answer question ONE and any TWO of the remaining
Symbols used bear the usual meaning.

KIBU observes ZERO tolerance to examination cheating

This Paper Consists of 5 Printed Pages. Please Turn Over.

You may find the following information useful:-

Speed of light in the vacuum, $c = 3.00 \times 10^8$ — ms^{-1}; Planck's constant, $h = 6.63 \times 10^{-34}$ J s

$\hbar = \frac{h}{2\pi} = 1.055 \times 10^{-34} J.s$;Mass of a proton $m_p = 1.66 \times 10^{-27}$ kg

Mass of the electron, $m_e = 9.11 \times 10^{-31}$ kg

1 eV $= 1.6 \times 10^{-19}$ J ; 1MeV $= 10^6$ eV ; 1nm $= 10^{-9}$ m.

$$\int \sin^2 bx\, dx = \frac{x}{2} - \frac{1}{4b}\sin(2bx) \qquad \int_{-\infty}^{+\infty} e^{-ax^2}\, dx = \sqrt{\frac{\pi}{a}} \qquad \int_{-\infty}^{+\infty} x.e^{-2ax^2}\, dx = 0$$

$$\int x^{2n} e^{-bx^2}\, dx = \frac{1\times 3\times \ldots\ldots \times (2n-1)}{2^{n+1}}\left(\frac{\pi}{b^{2n+1}}\right)^{\frac{1}{2}} \quad ; \; n = 1, 2, 3, \ldots\ldots \quad \sin^2 A = \frac{1}{2}[1 - \cos 2A]$$

QUESTION ONE [30 MARKS]

i. Write down the **time-dependent** and **time-independent** Schrodinger equation for the wave function $\Psi(x, t)$ of a particle of mass m moving in one dimension x in the potential V (x). [4 Marks]

ii. Write down the **normalization condition** for the wave function $\Psi(x)$ of a particle which is restricted to move in the interval $-\infty \leq x \leq +\infty$. [2Marks]

iii. A particle is constrained to move in the one-dimensional interval $-a \leq x \leq a$. Write down the definition of the **expectation value** $\langle x \rangle$ and $\langle x^2 \rangle$ of the coordinate x when the wave function of the particle is $\psi(x)$. [3 Marks]

iv. What is the quantum **mechanical interpretation** of ψ and $\psi\psi^*$, where ψ is a solution of the Schrödinger equation? Why does ψ have to be square-integrable? What does this mean in mathematical terms?

v. Find the normalization constant, N, for the wave function, [3 Marks]

$$\psi(x,t) = Ne^{-\frac{ax^2}{2}} e^{-\frac{i\varepsilon_0 t}{\hbar}}$$

For $-\infty \leq x \leq +\infty$ in the interval

Write down the normalized wave function. Use the standard integral

$$\int_{-\infty}^{+\infty} e^{-ax^2}\, dx = \sqrt{\frac{\pi}{a}}$$

vi. A one-particle, one-dimensional system has a normalized wavefunctions of the for

$$\psi_n(x) = \sqrt{\frac{2}{L}}.\sin\left(\frac{n\pi x}{L}\right)$$

at t = 0, where L = 1 nm. At t= 0, the particle's position is measured. Find the probability that the measured value lies between x = 0.1 nm and x = 0.2 nm. [2 Marks]

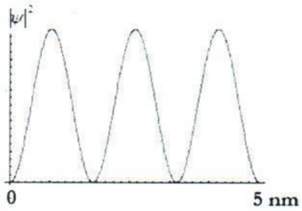
vii. Consider a particle in the ground state of an infinite potential box of length L. The wave functions of the particle in an infinite potential well are given by

$$\psi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$$

a. Find the probability density $|\psi|^2$ for the ground state. [1 Marks]

b. What is the probability of finding a particle in the interval between x = 0.50L and x = 0.51L (in the ground state) ? [2 Marks]

viii. Consider an electron trapped in a 1D well with L = 5 nm. Suppose the electron is in the following state:



Assume that the potential seen by the electron is approximately that of an infinite square well. What is the energy of the electron in this state (in eV)? [3 marks]

ix. Explain what is meant by the orthogonality of two wavefunctions $\psi_1(x)$ and $\psi_2(x)$, in the quantum mechanics of a particle on a line $-\infty < x < \infty$ [3 Marks]

x. An electron is acted by a potential V(x) within the region . Its wavefunction is given as

$$\psi(x,t) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right) e^{-i\omega t} \qquad 0 \le x \le a$$

Use the **time dependent** Schrodinger equation to show that the potential V(x) acting on
the electron is given by [4 marks]

$$V = \frac{-\hbar^2\pi^2}{2ma^2} + \hbar\omega$$

xi. For a 1-D system momentum operator $\hat{p}_x = -i\hbar \frac{d}{dx}$ and the position operator $\hat{x} = x$ *show that* $[\hat{p}_x , \hat{X}] = -i\hbar$.

[3 marks]

QUESTION TWO [20 MARKS]

a) Express the integral $\int \psi_n^*(x)\psi_m(x)dx = \delta_{nm}$ in terms of the Dirac's bra and ket notation [2 marks]

b) Write down the time independent eigen-value Schrodinger equation [2 marks]

c) A quantum system has a set of eigenstates $u_n(x)$, with energies E_n. The system is placed in a state ψ that is not an eigenstate; use the fact that the u_n are a complete set to show that the expectation value of the Hamilitonian, $\langle\psi|H|\psi\rangle$, always overestimates the ground-state energy. [10 marks]

d) A region of space has a potential step such that particles have a wave function given by

$$\psi(x,t) = \begin{cases} \frac{5a}{\sqrt{2}} e^{i(K_1x-Et/\hbar)} + \frac{3a}{\sqrt{2}} e^{i(-K_1x-Et/\hbar)}, & x < 0 \\ \frac{8a}{\sqrt{2}} e^{i(K_2x-Et/\hbar)}, & x > 0 \end{cases}$$

The incident particles, initially at $x << 0$, are initially travelling in the positive x direction.

(i) What fraction of the incident particles will be reflected ? [3 marks]

(ii) What is K_2 / K_1 ? [3 marks]

QUESTION THREE [20 MARKS]

Consider a particle of mass m inside a box of size L with infinite walls,

$$V(x) = \begin{cases} 0 & \text{for } 0 \le x \le L \\ \infty & \text{elsewhere} \end{cases}$$

The wave function is specified at $t = 0$ to be

$$\psi(x, t = 0) = C\left[3\sin\left(\frac{2\pi x}{L}\right) - 2\sin\left(\frac{3\pi x}{L}\right)\right]$$

a) Show that the above expression for □$(x,t = 0)$ can also be written as [3 marks]

$$\psi(x, t = 0) = C\sqrt{\frac{L}{2}}\left[3u_2(x) - 2u_3(x)\right],$$

given that the eigen-functions for a wave function in an infinite potential well is given by

$$u_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{2\pi x}{L}\right)$$

b) Determine the normalization coefficient C. [5 marks]

c) Expand the wave function at the initial time $\psi(x, t = 0)$ in terms of eigenfunctions of the infinite box, i.e. determine the expansion coefficients c_n [3 marks]

d) Write down $\psi(x,t)$, at an arbitrary later time t. [5 marks]

e) If a measurement of the particle's energy at time t is performed, what will be the possible outcomes, and with what probability will those values be measured? What is the average energy

$\langle E \rangle$ of the particle in the box? Is $\langle E \rangle$ changed by the measurement? [4 marks]

QUESTION FOUR [20 MARKS]

a) Use the normalization integral to show that the normalization constant $N=(2/L)^{1/2}$ and hence write down the normalized wave function for

$$\psi(x,t) = Ne^{-\frac{ax^2}{2}}e^{-\frac{i\varepsilon_0 t}{\hbar}},$$

in the same interval $-\infty \leq x \leq +\infty$ [4 marks]

b) Determine the expectation value $\langle x \rangle$ and $\langle x^2 \rangle$ and hence show that the uncertainty

$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \frac{1}{\sqrt{2a}}$ for the wave function described in (a) above [9 marks]

c) Determine the expectation value $\langle p \rangle$ and $\langle p^2 \rangle$ and hence show that the

Uncertainty in momentum, $\Delta p = \sqrt{\langle P^2 \rangle - \langle P \rangle^2} = \hbar\sqrt{\frac{a}{2}}$ [5 marks]

d) Hence demonstrate that $\Delta p \, \Delta x = \frac{\hbar}{2}$ and comment very briefly how this compares to the 2

uncertainty principle [2 marks]

QUESTION FIVE [20 MARKS]

a) Write down the time-independent Schrodinger equation for a particle in a one-dimensional harmonic oscillator potential, $V = \frac{m\omega^2 x^2}{2}$

[3 marks]

b) The ground-state wave function is of the form $\psi = A\exp(-\alpha x^2)$. Determine the constant□

, and hence the ground-state energy. [10 marks]

c) A particle of mass m is confined to a harmonic oscillator potential given by $V = m\omega^2 x^2/2$, where $\omega = K/m$ and K is the force constant. The particle is in a state described by the wave function

$$\psi(x,t) = A e^{\left(\frac{-m x^2 \omega}{2\hbar} - \frac{i\omega t}{2}\right)}$$

Verify that this is a solution of Schrodinger's equation. [7 marks]