(Knowledge for Development)

KIBABII UNIVERSITY

UNIVERSITY EXAMINATIONS

2021/2022 ACADEMIC YEAR

FIRST YEAR SECOND SEMESTER

MAIN EXAMINATION

FOR THE DEGREE OF MASTER OF SCIENCE IN PURE MATHEMATICS

COURSE CODE: **MAT 822**

COURSE TITLE: **ABSTRACT INTEGRATION II**

DATE: 23/05/2022 **TIME:** 9:00 AM - 11:00 AM

INSTRUCTIONS TO CANDIDATES

Answer any THREE Questions

TIME: 2 Hours

QUESTION ONE (20 MARKS)

a) Define the following terms

   i. Rectangle in $X \times Y$

   ii. Sections in $X \times Y$

b)

   i. State the lemma on monotone classes

   ii. State the unique extension theorem

   iii. State the monotone convergence theorem

c) Given ρ is a ring of subsets of X, $\mathcal{J}$ a ring of subsets of Y and let R be a ring generated by the class of all rectangles $E \times F$ where $E \in \rho$ and $F \in \mathcal{J}$, show that R coincides with the class of all finite disjoint unions $M = \cup_{i=1}^{n} E_i \times F_i$ where $E_i \in \rho$ and $F_i \in \zeta$

QUESTION TWO (20 MARKS)

a) Define the following terms

   i. The x-section of a mapping

   ii. The y-section of a mapping

b) Show that if M is any subset of $X \times Y$ then $(\chi_M)_x = \chi_{M_x}$ and $(\chi_M)^y = \chi_M y$ for all $x \epsilon X$ *and* $y \epsilon Y$

c) Given that h is measurable with respect to $\rho \times \mathfrak{I}$ show that for each $x \epsilon X$ the function h_x is measurable with respect to $\mathfrak{I}$ and for each $y \epsilon Y$ the function h^y is measurable with respect to ρ

QUESTION THREE (20 MARKS)

a) Given $G \in \rho$ and consider the measure space X, ρ, μ_G where μ_G is the contraction of μ by G, suppose that $f: X \rightarrow R$ is integrable with respect to μ_G and that $f = 0$ on $X - G$. show that f is also integrable with respect to μ and $\int f \, d\mu = \int f \, d\mu_G$

b) Suppose $M \subseteq P \times Q$ where M is in $\rho \times \mathfrak{I}$ and $P \times Q$ is a finite rectangle. Let $h = \chi_M$, show that;

   i. for each $x \epsilon X$ one has $h_x \in L'(\mu)$ and $\int h_x \, d\mu = f_M(x)$

   ii. for each $y \epsilon Y$ one has $h^y \in L'(\mu)$ and $\int h^y \, d\mu = g^M(y)$

iii. Moreover the functions h, f_M and g^M are integrable and $\pi(m) = \int h\, d\pi = \int f_M\, d\mu = \int g^M\, d\mu$

c) Suppose M is in $\rho \times \mathcal{J}$ show that the following conditions are equivalent

i. $\pi(M) < \infty$

ii. There exists an $f \in L'(\mu)$ such that $f_M = f$ a.e. $[u]$

iii. There exists a $g \in L'(v)$ such that $g^M = g$ a.e . $[v]$ in this case, $\int f\, d\mu = \int g\, dv = \pi(M)$

QUESTION FOUR (20 MARKS)

a) Define the following terms

i. Absolutely continuous function

ii. A signed measure

iii. Dominated measure

b) Given (X, ρ, μ) is a finite measure space and v is a finite measure on ρ such that $v \ll \mu$, show that there exists an $f \in L'(u)$ such that $v = u_f$ and $f \geq 0$ show also that if $g \in L'(v)$ such that $v = u_g$ then $g = f\ a.e.\,[u]$

QUESTION FIVE (20 MARKS)

a) Given v is a finite signed measure on ρ, Show that

i. $|v(E)| \leq |v|(E)$ for every measurable set E

ii. $v = 0$ if and only if $|v| = 0$

iii. $v^+ \ll |v|$ and $v^- \ll |v|$

iv. If μ is a measure on ρ then $|v| \ll \mu$ if and only if both $v^+ \ll u$ and $v^- \ll u$

b) Suppose (X, ρ, μ) is a $\sigma - finite$ measure space and v is a finite signed measure on ρ. Show that the following are equivalent

i. $v \ll \mu$

ii. $|v| \ll \mu$

iii. $v^+ \ll \mu$ and $v^- \ll \mu$

iv. v is AC with respect to μ

v. $|v|$ is AC with respect to μ

vi. $\mu(E) = 0$ implies $v(E) = 0$

vii. There exists an $f \epsilon l'(\mu)$ such that $v(E) = \int_E f d\mu$ for every measurable set E. In this case, f is unique almost everywhere $[\mu]$