(Knowledge for Development)

KIBABII UNIVERSITY

UNIVERSITY EXAMINATIONS

2021/2022 ACADEMIC YEAR

FIRST YEAR SECOND SEMESTER

MAIN EXAMINATION

FOR THE DEGREE OF MASTER OF SCIENCE IN PURE MATHEMATICS

COURSE CODE: **MAT830**

COURSE TITLE: **REPRESENTATION THEORY OF GROUPS**

DATE: 24/05/2022 **TIME: 9 AM -12 AM**

INSTRUCTIONS TO CANDIDATES

Answer Any THREE Questions

TIME: 3 Hours

This Paper Consists of 5 Printed Pages. Please Turn Over.

QUESTION ONE (20MARKS)

a) Let G be a finite group and F field. Explain what is meant by the following terms;

i. A representation of G over F

ii. An irreducible representation

iii. The degree of a representation

iv. Decomposable representation

b) Using the natural basis { v_1, v_2, v_3 } write down the 3-dimensional permutation representation of $G \cong S_3$ (It is enough to give the matrices on the generators of the group)

c) State Marschke's theorem

d) For this question, G is a group of order 20 with 5 conjugacy classes. Here are the first two lines of its character table

	I	g_2	g_3	g_4	g_5
|ccl|	1	4	5	5	5
χ_1	1	1	1	1	1
χ_2	1	1	i	-1	-i

i. What are the dimensions (degrees) of the remaining representations of G? justify .

ii. Find another 1-dimensional representation of G

QUESTION TWO (20MARKS)

a) Given T as a representation;

   i. Define a T-invariant subspace

   ii. Show that kerT(g) is T-invariant for all $g \in G$.

b) Let G be a finite group with two representations T and T^I on V.

   iii. Define what is meant by T is equivalent to T^I

   iv. Show that if S and T are equivalent F-representations with characters χ and ψ respectively then $\chi = \Psi$.

c) Let G be any finite group, $G = \{g_1, \ldots\ldots\ldots, g_n\}$ and V a vector space over a field F. Choose an ordered basis $\{v_1, \ldots\ldots\ldots, v_n\}$ and define $T(g_i)v_j = vk$ if $g_i g_j = gk$. Show that T is a representation of G over V.

d) Let $G = S_3$ and $X = \{1,2,3\}$.

   i. Find the (matrix) permutation representation of S_3 (enough to give the matrices of the of the generators.

   ii. What is the degree of this representation?

QUESTION THREE (20MARKS)

a) Define the term character of a representation T.

b) Show that characters are class functions

c) Suppose that the char $F \neq |G|$. If χ_1 and χ_2 are F- characters of G. Define the inner product $< \chi_1, \chi_2 >$

d) Compute the character table of S_3 the symmetric group.

e) Consider the class function $\phi: S_3 \rightarrow \mathbb{C}$ defined by $\phi(e) = 4$, $\phi(12) = 0$, $\phi(123)=$ -5. write down ϕ as a linear combination of irreducible characters of S_3.

QUESTION FOUR(20MARKS)

a) Let G be a group acting on a set X. Define a transitive action.

State the orbit-stabilizer theorem

b) Let G be the alternating group $A_4 \subset S_4$. There are 4 conjugacy classes in A_4 with representations I, (123), (132), (12)(34) and sizes 1,4,4,3 respectively.

i. Find the number of irreducible representations and their dimensions (degrees).

ii. $\chi_4 : G \to \mathbb{C}$ be the function which is constant on each cycle type and takes the values $\chi_4(1) = 3$, $\chi_4(123) = \chi_4(132) = 0$ and $\chi_4((12)(34)) = -3$. Using the permutation character, show that χ_4 is a char of A_4. Prove that it is irreducible.

iii. The character table now looks like

	1	(123)	(132)	(12)(34)
	1	4	4	3
χ_1	1	1	1	1
χ_2	3	0	0	-1
χ_3	1	a	b	c
χ_4	1	d	e	f

Use orthogonality relations to show that c = f. i.e find the values of c & f.

QUESTION FIVE (20MARKS)

a) What is meant by the character table of a group.

b) State the row and column orthogonality relations for a character of a representation

c) Let $\varphi = \{\pm 1, \pm i, \pm j, \pm k\}$ be the quarternion group of order 8 with the usual relations $-1^2 = 1$, $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = I$, $ki = -ik = j$. φ has 5 conjugacy classes viz 1, -1, $\{I, -I\}$, $\{j, -j\}$ and $\{k, -k\}$. Compute the character table of φ. Find the centre of the group φ.

d) There is a representation of φ. $\rho: \varphi \to GL_2(\mathbb{C})$ defined by $\rho(i) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\rho(j) = \begin{pmatrix} \xi & 0 \\ 0 & \xi \end{pmatrix}$

i. Compute $\rho(-1)$ and $\rho(k)$

ii. Show that ρ is irreducible .