(Knowledge for Development)

# KIBABII UNIVERSITY

## UNIVERSITY EXAMINATIONS

## 2021/2022 ACADEMIC YEAR

## FORTH YEAR FIRST SEMESTER

## MAIN EXAMINATION

## FOR DEGREE OF BACHELOR OF SCIENCE MATHEMATICS

**COURSE CODE:** **MAP 413**

**COURSE TITLE:** **FUNCTIONAL ANALYSIS**

**DATE:** 17/05/2022 **TIME:** 9:00 AM - 11:00 AM

## INSTRUCTIONS TO CANDIDATES

Answer question ONE and any other two questions

TIME: 2 Hours

**QUESTION ONE (30 MARKS)**

a) Define a metric space

b) Show that the three-dimensional space $\mathbb{R}^3$with the distance

$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}$ is a metric space

c) Show that the space $(\mathcal{L}^\infty, d)$ with d defined by $d(x, y) = sup_{j\epsilon N} |x_i - y_j|$ is a metric space

**QUESTION TWO (20 MARKS)**

a) Show that the following

i. Holder's inequality for sums $\sum_{j=1}^{\infty}|x_j y_j| \le (\sum_{j=1}^{\infty}|x_j|^p)^{\frac{1}{p}} (\sum_{j=1}^{\infty} |y_j|^q)^{\frac{1}{q}}$

ii. Minkowski inequality for sums $(\sum_{j=1}^{\infty}|x_j + y_j|^p)^{\frac{1}{p}} \le (\sum_{j=1}^{\infty}|x_j|^p)^{\frac{1}{p}} + (\sum_{j=1}^{\infty}|y_j|^p)^{\frac{1}{p}}$

b) Show that the sequence space $\mathcal{L}^p$ with d defined by $d(x, y) = (\sum_{j=1}^{\infty}|x_j - y_j|^p)^{\frac{1}{p}}$ is a metric space

**QUESTION THREE (20 MARKS)**

a) Show that the space $\mathcal{L}^\infty$ is not separable

b) Show that the space $\mathcal{L}^p$ $1 \le p \le \infty$ is separable

c) Show that a subspace M of a complete metric space X is itself complete if and only if the set M is closed in X

**QUESTION FOUR (20 MARKS)**

a) Show that the Euclidean space $\mathbb{R}^n$ is complete

b) Define the following terms

i. Normed space

ii. Banach space

c) Show that a metric d induced by a norm on a normed space X satisfies

i. $d(x + a, y + a) = d(x, y)$

ii. $d(ax, ay) = |a|d(x, y)$

**QUESTION FIVE (20 MARKS)**

a) Given $\{x_1, x_2, \dots, x_n\}$ are a linearly independent set of vectors in a anormed space X (of any dimension) show that there is a number $c > 0$ such that for every choice of scalars $\propto_1, \dots, \propto_n$ we have $||\propto_1 x_1 + \cdots + \propto_n x_n|| \geq c(|\propto_n| + \cdots + |\propto_n|)$

b) Show that a compact subset M of a metric space is closed and bounded

c) Define a linear operator T