(Knowledge for Development)

KIBABII UNIVERSITY

UNIVERSITY EXAMINATIONS

2020/2021 ACADEMIC YEAR

FIRST YEAR SECOND SEMESTER

MAIN EXAMINATION

FOR THE DEGREE OF MASTER OF SCIENCE IN PURE MATHEMATICS

COURSE CODE: **MAT 822**

COURSE TITLE: **ABSTRACT INTEGRATION II**

DATE: 13/10/21 **TIME:** 9.00AM - 12.00 pm

INSTRUCTIONS TO CANDIDATES

Answer Any other THREE Questions

TIME: 2 Hours

QUESTION ONE (20 MARKS)

a) State the following theorems

   i. The lemma on monotone classes, LMC

   ii. The unique extension theorem, UET

   iii. The monotone convergence theorem, MCT

b) Given ρ is a ring of subsets of X, ζ a ring of subsets of Y and let R be a ring generated by the class of all rectangles $E \times F$ where $E \in \rho$ and $F \in \zeta$, show that R coincides with the class of all finite disjoint unions $M = \cup_{i=1}^{n} E_i \times F_i$ where $E_i \in \rho$ and $F_i \in \zeta$

c) Define the following terms

   i. Measurable rectangle

   ii. Cartesian product of measurable spaces

   iii. X-section of M in $X \times Y$

   iv. Y-section of M in $X \times Y$

QUESTION TWO (20 MARKS)

a) Show that for each finite rectangle $P \times Q$ there exists a unique(finite) measure $T^{P\times Q}(E \times F) = \mu(P \cap Q)\upsilon(Q \cap F)$ for every measurable rectangle $E \times F$.

b) Show that if $P_1 \times Q_1 \subseteq P_2 \times Q_2$ then $T^{P_1\times Q_1} \leq T^{P_2\times Q_2}$

c) Show that if (X, ρ, μ) and (Y, τ, υ) are arbitrary measure spaces, there exists a unique measure π on $\rho \times \tau$ having the following properties

   i. $\pi(P \times Q) = \mu(P)\upsilon(Q)$ for every finite rectangle $P \times Q$

   ii. $\pi(M) = LUB\{\pi(P \times Q) \cap M, p \in \rho_\infty, q \in \tau_\varphi\}$ for each M in $\rho \times \tau$

QUESTION THREE (20 MARKS)

a) **Given** (X, ρ, μ) is a measure pace and $\boldsymbol{\upsilon}$ is a finite measure on ρ, show that the following conditions on $\boldsymbol{\upsilon}$ are equivalent

   i. υ is absolutely continuous with respect to μ

   ii. $\mu(E) = 0$ implies $\upsilon(E) = 0$

b) Suppose $\boldsymbol{v}$ is a finite signed measure on ρ. Let E, F and $E_n (n = 1,2,3 \ldots)$ be measurable sets, show that

i. $v(\phi) = 0$

ii. v is finitely additive

iii. v is subtractive; if $E \subseteq F$, then $v(F - E) = v(E) - v(F)$

iv. If the E_n are mutually disjoint, the series $\sum_1^\infty v(E_n)$ converges absolutely

v. If $E_n \uparrow E$, then $v(E_n) \to v(E)$

vi. If $E_n \downarrow E$, then $v(E_n) \to v(E)$

vii. Let $(E_i)_{i\in I}$ be a family of measurable sets such that $E_i \cap E_j = \emptyset$ when $i \neq j$. If $v(E_i) > 0$ for all i then I is countable. Same conclusion if $v(E_i) < 0$ for all i. Same conclusion if $v(E_i) \neq 0$ for all i.

QUESTION FOUR (20 MARKS)

a) Define the following terms. Given A a locally measurable set, define

i. Purely positive

ii. Purely negative

iii. Equivalent to zero

b) Let v be a finite signed measure on ρ and let E be a measurable set

i. Show that if $v(E) > 0$, there exists a measurable set E_o such that $E_o \subseteq E$, $E_o \geq 0$ and $v(E_o) > 0$

ii. Show that if $v(E) < 0$, there exists a measurable set E_o such that $E_o \subseteq E$, $E_o < 0$ and $v(E_o) < 0$

c) Suppose v *and* μ are finite measures on ρ such that $v < \mu$ and $v \neq 0$ show that there exists an $\in > 0$ and a measurable set E such that

i. $E \geq 0$ with respect to finite signed measure $v - \epsilon\mu$

ii. $(v - \epsilon\mu)(E) > 0, v(E) > 0, \mu(E) > 0$

QUESTION FIVE (20 MARKS)

a) Suppose (X, ρ, μ) is a $\sigma - finite$ measure space and υ is a finite signed measure on ρ. Show that the following are equivalent

i. $\upsilon \ll \mu$

ii. $|\upsilon| \ll \mu$

iii. $\upsilon^+ \ll \mu$ and $\upsilon^- \ll \mu$

iv. ν is absolutely continuous with respect to μ

v. $|\upsilon|$ is AC with respect to μ

vi. $\mu(E) = 0$ implies $\upsilon(E) = 0$

vii. There exists an $f \epsilon l'(\mu)$ such that $\upsilon(E) = \int_E f d\mu$ for every measurable set E. In this case, f is unique almost everywhere $[\mu]$

b) Given (X, ρ, μ) is a finite measure space, and suppose φ is a positive linear form on $L'(\mu)$, that is φ is a real-valued function defined on $L'(\mu)$ such that

1) $\varphi(f_1 + f_2) = \varphi(f_1) + \varphi(f_2)$
2) $\varphi(cf) = c\varphi(f)$
3) $\varphi(f) \geq 0$ whenever $f \geq 0$

Assume, moreover that φ is bounded, that is assume there exists a real number $M \geq 0$ such that $|\varphi(f)| \leq M \int |f| d\mu$ for all f in $L'(\mu)$. Show that there exists a bounded measurable function $g, g \geq 0$ such that $\varphi(f) = \int gf d\mu$ for all f in $L'(\mu)$.