(Knowledge for Development)

KIBABII UNIVERSITY

UNIVERSITY EXAMINATIONS

2020/2021 ACADEMIC YEAR

FOURTH YEAR SECOND SEMESTER

MAIN EXAMINATION

FOR THE DEGREE OF BACHELOR OF SCIENCE

COURSE CODE: MAT 404

COURSE TITLE: DIFFERENTIAL TOPOLOGY

DATE: 4/10/2021 **TIME:** 9:00 AM - 11:00 AM

INSTRUCTIONS TO CANDIDATES

Answer Question One and Any other TWO Questions

TIME: 2 Hours

This Paper Consists of 3 Printed Pages. Please Turn Over.

QUESTION ONE (30 MARKS)

a). Define the following terms

(i). n-dimensional manifold (2 mks)

(ii). Tangent space (2 mks)

(iii). Diffeomorphism (2 mks)

(iv). Regular Value (2 mks)

(v). Proper map (2 mks)

b). Prove that the function $f: \mathbb{R} \to \mathbb{R}$ defined by $x \mapsto x^3$ is a homeomorphism but not smooth. (4 mks)

c). Prove that a subset of $\mathbb{R}^n$ is a manifold (4 mks)

d). Show that the map $f(t) = (\cos t, \sin t)$ is a local diffeomorphism but not a global diffeomorphism. (4 mks)

e). Given that $f: X \to Y$ is a smooth map with regular value $y \in Y$, prove that $f^{-1}(y)$ is a submanifold of X. (4 mks)

f). If the smooth map $f: X \to Y$ is transversal to a submanifold $Z \subset Y$, prove that the pre-image $f^{-1}(Z)$ is a submanifold of X. (4 mks)

QUESTION TWO (20 MARKS)

a). Differentiate between immersion and submersion (4 mks)

b). Let Z be a pre-image of a regular value $y \in Y$ under the smooth map $f: X \to Y$. Then the kernel of the derivative $df_x: T_x(X) \to T_y(Y)$ at any point $x \in Z$ is precisely the tangent space Z, $T_x(Z)$. (5 mks)

b). A cone is not a manifold. Explain. (3 mks)

c). Define the term an embedding hence prove that an embedding $f: X \to Y$ maps X diffeomorphically into a submanifold of Y. (8 mks)

QUESTION THREE (20 MARKS)

a). Consider a unit circle $x^2 + y^2 = 1$.

(i). Define the term chart hence suitably define four charts on the unit circle that covers its. (4 mks)

(ii). Let $T: (0,1) \to (0,1)$ be a transition map. Define the map T in terms of two charts in (i) above hence determine $T(a)$ for $a \in (0,1)$. (3 mks)

b). Prove that the dimension of the tangent space $T_x(X)$ is equal to that of the manifold X. (5 mks)

c). Determine the tangent space to the paraboloid defined by $x^2 + y^2 - z^2 = a$ at $(\sqrt{a}, 0,0)$.

(8 mks)

QUESTION FOUR (20 MARKS)

a). Consider a circle $S' = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.

(i). Define two parametrizations ϕ_1 and ϕ_2 that maps the circle to the subset $(-1,1)$ of the x axis. (2 mks)

(ii). Show that ϕ_1 is locally invertible and its inverse is a projection on x –axis. (2 mks)

b). (i). When is a function $f: X \to Y$ transversal to a submanifold $Z \in Y$? (2 mks)

(ii). For which values of a does the hyperboloid $x^2 + y^2 - z^2 = 1$ intersect the sphere $x^2 + y^2 + z^2 = a$ transversally? What does the intersection look like for different values of a? (5 mks)

c). State the prove the Sard's theorem. (6 mks)

d). Let $f: X \to Y$ be a smooth function. Define the critical values of f, hence use Sards theorem, determine its measure (3 mks)

QUESTION FIVE (20 MARKS)

a). Define the differential df_x hence express it in terms of the derivative of two parametrizations to its domain and codomain. (4 mks)

b). Given that $X \overset{f}{\rightarrow} Y$ and $Y \overset{g}{\rightarrow} Z$ are smooth maps of manifolds. Prove that $d(g \circ f)_x = dg_{f(x)} \circ df_x$. Use commutive diagrams for Illustration. (6 mks)

c). Let $S^1 \subset \mathbb{C}$ be the set $\{z \in \mathbb{C} : |z| = 1\}$. Define a map $F: S^1 \to S^1$ by $z \mapsto z^2$ where $z = x + iy$ for $i = \sqrt{-1}$. Determine the differential df at i. (10 mks)