(Knowledge for Development)

KIBABII UNIVERSITY

UNIVERSITY EXAMINATIONS

2020/2021 ACADEMIC YEAR

FIRST YEAR FIRST SEMESTER

MAIN EXAMINATION

FOR THE DEGREE OF MASTER OF SCIENCE IN APPLIED MATHEMATICS

COURSE CODE: **MAT 851**

COURSE TITLE: **ODE I**

DATE: 12/10/21 **TIME: 9 AM -12 AM**

INSTRUCTIONS TO CANDIDATES

Answer Any THREE Questions
Show all the necessary working

TIME: 3 Hours

This Paper Consists of 2 Printed Pages. Please Turn Over.

QUESTION1 [20 MARKS]

Given the system of first order ordinary differential equations

$$\frac{dx}{dt} = 5x + y + 3z + 1$$

$$\frac{dy}{dt} = x + 7y + z$$

$$\frac{dz}{dt} = 3x + y + 5z$$

(i) Express the system in the matrix form $\dot{\underline{X}} = A\underline{X} + \underline{f}$

(ii) Show that $\underline{u} = [1,1,1]^t$, $\underline{v} = [-1,0,1]^t$, $\underline{v}_2 = [1,-2,1]^t$ are eigenvectors of A

(iii) Determine $\Phi(t)$, the fundamental matrix of the system

Obtain $\underline{X}$ the general solution of the system [20 marks]

QUESTION2 [20 MARKS]

i)For the $L-R-C$ electrical network in figure1 below, show that the current $i_1(t)$ and charge $q(t)$ satisfies the system of differential equations

$$\frac{d}{dt}\begin{pmatrix} i_1 \\ q \end{pmatrix} = \begin{pmatrix} \frac{R_1 R_2}{L(R_1+R_2)} & \frac{R_1}{CL(R_1+R_2)} \\ \frac{R_1}{(R_1+R_2)} & \frac{1}{C(R_1+R_2)} \end{pmatrix}\begin{pmatrix} i_1 \\ q \end{pmatrix} + \begin{pmatrix} \frac{E}{L} \\ 0 \end{pmatrix}.$$

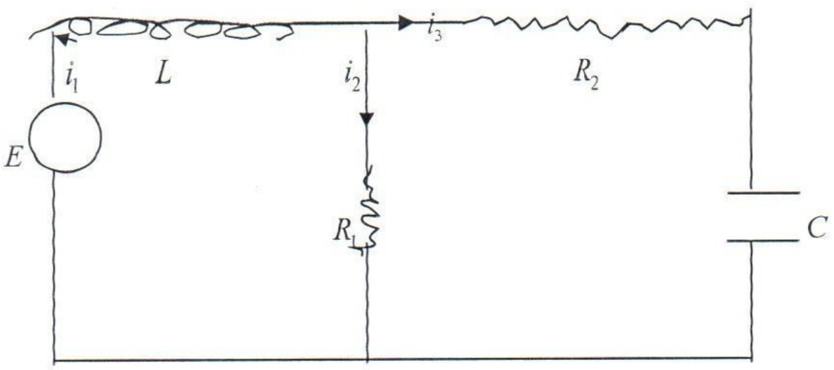

Fig 1

ii) Obtain a simplified matrix equation of the form $\dot{\underline{X}} = A\underline{X} + \underline{f}$ representing system in part i) above. Discuss conditions under solution is degenerate.

iii) Use variation of parameters to solve the system if

$\dot{\underline{X}} = A\underline{X} + \underline{f}$ $R_1 = 500\Omega$, $R_2 = 100\Omega$, $L = 15h$, $C = 0.054f$ and $E(t) = 110V$

QUESTION3 [20 MARKS]

(a) (i) Find the Laplace transform: $f(t) = t^2 e^{-40t} \sin 3t$

(ii) Evaluating $L^{-1}\left[\frac{1}{s^2(s^2+4)}\right]$ [8 marks]

(b) Solve the system of linear ordinary differential equations

$\frac{dx}{dt} - y = e^t, \; \frac{dy}{dt} + x = \sin t \; ; \; x(0) = 1, y(0) = 0$ [12 marks]

QUESTION4 [20 MARKS]

Given the system of nonlinear differential equations

$$\frac{dx}{dt} = -2xy$$

$$\frac{dy}{dt} = -x + y + xy - y^2$$

(a)Find all its critical points [8 marks]

(b) Determine the stability nature of each of the critical points in part (a) [12 marks]

QUESTION5 [20 MARKS]

(a) If $A = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}$ compute the exponential matrix e^{At}. Deduce the solution of the linear system

$\underline{X}' = A\underline{X} \quad subject \;\; to \quad \underline{X}(0) = \underline{I}$ [7 marks]

(b)i) Show that $\left[e^{At}\right]_{t=0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

)ii) Verify that $\left[e^{At}\right]^{-1} = \left[e^{A(-t)}\right]$

ili) Show that $\left[e^{At}\right] = \phi(t)\left\{\phi^{-1}(t)\right\}_{t=0}$ where $\phi(t)$ is the fundamental matrix of the linear system

[13 marks]

LAPLACE TRANSFORMS

$f(t)$	Laplace transform of $f(t)$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
e^{at}	$\frac{1}{s-a}$
$\cos bt$	$\frac{s}{s^2+b^2}$
$\sin bt$	$\frac{b}{s^2+b^2}$
$e^{-at}\sin bt$	$\frac{b}{(s+a)^2+b^2}$
$e^{-at}\cos bt$	$\frac{(s+a)}{(s+a)^2+b^2}$
$e^{-at}t^n$	$\frac{\Gamma(n+1)}{(s+a)^{n+1}}$ $n>-1$
t^n	$\frac{n!}{s^{n+1}}$
$e^{-at}t^n$	$\frac{n!}{(s+a)^{n+1}}$
$\frac{dy}{dt}$	$sY-y_0$; $Y=L(y)$
$\frac{d^2y}{dt^2}$	$s^2Y-sy_0-y_0'$; $Y=L(y)$
$\frac{d^ny}{dt^n}$	$s^nY-s^{n-1}y_0-s^{n-2}y_0-s^{n-3}y_0-\ldots$ $-sy^{(n-2)}(0)-y^{(n-1)}(0)$
u_t	$sU(x,s)-u(x,0)$; $U(x,s)=L[u(x,t)]$
u_{tt}	$s^2U(x,s)-su(x,0)-u_t(x,0)$;
u_{x^m}	$\frac{d^m}{dx^m}(U(x,s))$
u_{xt}	$s\frac{d}{dx}U(x,s)-\frac{d}{dx}u(x,0)$
$L[t\cos bt] = \frac{s^2-b^2}{(s^2+b^2)^2}$	

$$L[t\sin bt] = \frac{2bs}{\left(s^2+b^2\right)^2}$$

$$L\left[t^n f(t)\right] = (-1)^n \frac{d^n}{dt^n}\left[F(s)\right] : L\{f(t)\} = F(s)$$

LAPLACE TRANSFORMS

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$$L\left[J_0(t)\right] = \frac{1}{\sqrt{s^2+1}}$$

$$L\left[\frac{f(t)}{t}\right] = \int_t^{\infty} F(s)ds : L\{f(t)\} = F(s)$$

$$L^{-1}\left[L\{f(t)\}L\{g(t)\}\right] = \int_0^t f(\lambda)g(t-\lambda)d\lambda ,$$

$$L^{-1}\{\phi(s)\} = e^{-at}L^{-1}\{\phi(s-a)\} , L\{e^{-at}f(t)\} = L\{f(t)\}_{s\to s+a}$$

$J_0(t)$ is the Bessel function of order zero.