(Knowledge for Development)

KIBABII UNIVERSITY

UNIVERSITY EXAMINATIONS

2020/2021 ACADEMIC YEAR

THIRD YEAR SECOND SEMESTER

MAIN EXAMINATION

FOR THE DEGREE OF BACHELOR OF SCIENCE MATHEMATICS

COURSE CODE: MAP 321

COURSE TITLE: REAL ANALYSIS III

DATE: 12/10/2021 **TIME:** 9:00 AM - 11:00 AM

INSTRUCTIONS TO CANDIDATES

Answer Question One and Any other TWO Questions

TIME: 2 Hours

QUESTION ONE COMPULSORY (30 MARKS)

a) Define the following

i. Riemann-Stieltjes integral (3marks)

ii. Bounded variation (3marks)

iii. Uniform convergence (2marks)

iv. Finite series (2marks)

b) Show the function f defined by $f(x) = \begin{cases} 0 & if\ x\ is\ irrational \\ 1 & if\ x\ is\ rational \end{cases}$ is not of bounded variation on any interval. (8marks)

c) Suppose that $\{f_n\}$ is a sequence of continuous functions which converges uniformly to f on I, show that f is continuous (4marks)

d) Show that if n is an integer and x a positive number, then $\text{In}x^n = n\ \text{In}x$ (4marks)

e) Show that the series $1 + r + r^2 + \cdots + r^N = \frac{r^{N-1} - 1}{r-1}$ (4marks)

QUESTION TWO (20 MARKS)

a) Define the following

i. Partition (2marks)

ii. Lebesque integral (3marks)

iii. Integrable function (2marks)

b) Show that assuming F is an increasing step function on I so that

$$F(t) = \sum_{i=1}^{N} a_i I\{t \leq t_i\}$$

with $t_0 = \min(I) < t_1 < t_2 \ldots < t_N = \max(I)$ and $a_i \geq 0$ and if g is continuous, then $\int g(x)dF(x) = \sum_i^N g(t_i)a_i$ (5marks)

c) Suppose $f: [a, b] \to \mathbb{R}$ is a function, let $\{x_i | 0 \leq i \leq n\}$ be a partition of [a, b] and let $\{y_i | 0 \leq i \leq m\}$ be a partition of [a, b] such that $\{x_i | 0 \leq i \leq n\} \subseteq \{y_i | 0 \leq i \leq m\}$.

Show that $\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \leq \sum_{i=1}^{m} |f(y_i) - f(y_{i-1})|$ (8marks)

QUESTION THREE (20 MARKS)

a) State the monotone convergence theorem (4marks)

b) Suppose f_n and f are functions defined on an interval J. If there exists a sequence (x_n) in J such that $|f_n(x_n) - f(x_n)| \nrightarrow 0$, prove that (f_n) does not converge uniformly to f on J. (5marks)

c) Show that if f is increasing on $[a, b]$, then f is of bounded variation on $[a, b]$ and $V(f, [a, b]) = f(b) - f(a)$ (4marks)

d) Show that the logarithm of a product of two positive numbers is the sum of their logarithm (4marks)

e) Show that the infinite series $\sum_{n\geq 0} x^n$ converges if $|x| < 1$ and diverges if $|x| > 1$ (3marks)

QUESTION FOUR (20 MARKS)

a) Define the following
   i. Natural logarithm (2marks)
   ii. Convergent series (2marks)
   iii. Exponential function (2marks)

b) Suppose f_n is a sequence of continuous functions defined on an interval J which converges uniformly to a function f, show that f is continuous on J (5marks)

c) If f is a bounded function defined on [a; b] such that f is Riemann integrable, then f is Lebesgue integrable and $(R)\int_a^b f(x)\,dx = \int_{[a,b]} f(x)dx$ (5marks)

d) Shoe that the exponential of a sum is the product of the exponentials, that is $exp(a + b) = expa\ expb$ (4marks)

QUESTION FIVE (20 MARKS)

a) Suppose (f_n) is a sequence of continuous functions defined on an interval [a,b] which converges uniformly to a function f on [a,b] then show that f is continuous and $\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$ (6marks)

b) Assume that the sequence $\{f_n\}$ converges uniformly to g, show that $\{f_n\}$ converges pointwise and that $f = g$ (5marks)

c) State the Fatou's lemma (4marks)

f) For $x \in \mathbb{R}$, the infinite geometric series $\sum_{n\geq 0} x^n$ converges if $|x| < 1$ and $\sum_{n\geq 0} x^n = \frac{1}{1-x}$ then $a_n \to 0$ as $n \to \infty$ (5marks)