(Knowledge for Development)

KIBABII UNIVERSITY

UNIVERSITY EXAMINATIONS

2019/2020 ACADEMIC YEAR

FOURTH YEAR SPECIAL/SUPPLEMENTARY EXAMINATION

FOR THE DEGREE OF BACHELOR OF SCIENCE

COURSE CODE: **MAT 424**

COURSE TITLE: ODE III

DATE: 01/02/2021 **TIME:** 11 AM -1 PM

INSTRUCTIONS TO CANDIDATES

Answer Question One and Any other TWO Questions

TIME: 2 Hours

QUESTION ONE (30 MARKS)

a) Solve the initial value problem $\dot{x} = \beta x \quad x(0) = x_0$ using Picard's method of successive approximation. (5 Marks)

b) Investigate the stability of the second order equation

$$\ddot{x} + \dot{x}^3 + x = 0$$

At the origin of its phase plane (5 Marks)

c) Show that the system

$$\dot{x}_1 = -x_2 + x_1(1 - (x_1^2 + x_2^2)^{\frac{1}{2}})$$
$$\dot{x}_2 = x_1 + x_2(1 - (x_1^2 + x_2^2)^{\frac{1}{2}})$$

Has a limit cycle given by $x_1^2 + x_2^2 = 1$ (6 Marks)

d) Prove that every fundamental matrix solution $X(t)$ of $\dot{x} = Ax$ has the form where

$$X(t) = P(t)e^{Bt}$$

Where $P(t) = P(t+T)$ for all $t \in \mathbb{R}$, is a non-singular matrix and B is also an $n \times n$ constant matrix. (5 Marks)

e) Consider the initial value problem

$$\dot{x} = f(t,x) \quad x(t_0) = x_0 \quad t \in I \qquad (1)$$

Where $f \in C(U, \mathbb{R}^{n+1})$ U an open subset of $\mathbb{R}^{n+1}$ and I is in $\mathbb{R}$. Prove that (1) is equivalent to

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds \qquad (2)$$

And $x(t)$ is a solution of Equation (1) if and only if it is a solution of Equation (2) (4 Marks)

f) Prove the following ; If $g(t)$ is continuous real valued function that satisfies

$g(t) \geq 0$ and

$$g(t) \leq c + k\int_{t_0}^{t} g(s)ds \qquad t \in [0, a]$$

Where c and k are positive constants. It then follows that for all $t \in [0, a]$

$$g(t) \leq ce^{kt}$$

(5 Marks)

QUESTION TWO (20 MARKS)

Consider the nonlinear system

$$\dot{x} = -x + x(r^4 - 3r^2 + 1) \qquad (4)$$

$$\dot{y} = x + y(r^4 - 3r^2 + 1)$$

Where $r^2 = x_1^2 + x_2^2$

a) Use the Poincare Bendixson theorem to show that (4) has a periodic orbit in the annular region $D_1 = \{x \in \mathbb{R}^2 \,|\, 1 < |x| < 2\}$ (8 Marks)

b) Show that the origin is unstable focus for this system and use the Poincare Bendixson Theorem to show that there is periodic orbit in the annular region $D_2 = \{x \in \mathbb{R}^2 \,|\, 1 < |x| < 2\}$ (6 Marks)

c) Find the unstable and stable limit cycles of this system (6 Marks)

QUESTION THREE (20 MARKS)

a) Define the following terms (2 Marks)

(i) Equilibrium solution

(ii) Stability

b) Prove that the function $V(y_1, y_2) = y_1^2 + y_1^2 y_2^2 + y_2^4 \qquad (y_1, y_2) \in \mathbb{R}^2$

Is a strict Liapunov function for the system

$$\dot{x}_1 = 1 - 3x_1 + 3x_1^2 + 2x_2^2 - x_1^3 - 2x_1x_2^2$$

$$\dot{x}_2 = x_2 - 2x_1x_2 + x_1^2x_2 - x_2^3$$

At fixed point (1,0) (5 Marks)

c) Show that the phase portrait of

$$\ddot{x} - (1 - 3x^2 - 2\dot{x})\dot{x} + x = 0$$

Has a limit cycle (5 Marks)

d) Find the derivative of the function

$$f(x) = \begin{pmatrix} x_1 - x_2^2 \\ -x_2 + x_1x_2 \end{pmatrix} = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$$

And evaluate it at the point $x_0 = (1, -1)^T$ (4 Marks)

e) Let E be an open subset of $\mathbb{R}^n$ and $f: E \to \mathbb{R}^n$. Proof that if $f \in C^1(E)$, f is locally Lipschitz on E. (4 Marks)

QUESTION FOUR (20 MARKS)

Consider the differential equations that model the populations $x_1(t)$ and $x_2(t)$ at time $t \geq 0$ of two competing species

$$\dot{x}_1 = ax_1(1-x_1) - bx_1x_2 \qquad (3)$$
$$\dot{x}_2 = cx_2(1-x_2) - dx_1x_2$$

Let $a=1, b=2, c=1$ and $d=3$

(i) On one phase plane sketch the isoclines of the differential equations (3) and determine all its equilibriums (4 Marks)

(ii) Determine the type of stability of all equilibrium points in (i) above (6 Marks)

(iii) Sketch the phase plane and clearly indicate the direction of the vector field defined by (3) (2 marks)

(iv) State algebraically and sketch by shading appropriately the basin of attraction of each attracting fixed point. (4 Marks)

(v) What are the likely populations of the species in the long term. State the reasons for the choice of your answer. (2 Marks)

(vi) If a=3, b=2, c=4 and d=3. Show that the populations co-exist at some point $\bar{x}\left(\frac{2}{3}, \frac{1}{2}\right)$ (2 Marks)

QUESTION FIVE (20 MARKS)

For the system

$$\dot{x} = y$$
$$\dot{y} = x^2 - \mu$$

Where μ is a parameter.

a) Verify that this system is Hamiltonian and with the Hamiltonian

$$H(x,y) = \frac{y^2}{2} - \frac{x^3}{3} + \mu x$$ (4 Marks)

b) Show that for $\mu \geq 0$ the system has equilibrium points at $(x,y) = (\pm\sqrt{\mu}, 0)$ and no equilibrium points when $\mu < 0$ (so $\mu = 0$ is a bifurcation value of the parameter) (3 Marks)

c) Linearize the system at each of the equilibrium points and determine the behaviour of the solutions near the equilibrium points (4 Marks)

d) Sketch the level curves H (and hence the phase plane of the system) for $\mu \in \{-1, -0.5, 0.5, 1\}$ (5 Marks)

e) Describe the bifurcation that takes place at $\mu = 0$ (4 Marks)