(Knowledge for Development)

KIBABII UNIVERSITY

UNIVERSITY EXAMINATIONS

2020/2021 ACADEMIC YEAR

FIRST YEAR FIRST SEMESTER

MAIN EXAMINATION

FOR THE DEGREE OF MASTER OF SCIENCE IN PURE MATHEMATICS

COURSE CODE: **MAT 813**

COURSE TITLE: FUNCTIONAL ANALYSIS I

DATE: 21/07/21 **TIME:** 9 AM -12 AM

INSTRUCTIONS TO CANDIDATES

Answer Any THREE Questions

TIME: 3 Hours

This Paper Consists of **3 Printed Pages**. Please Turn Over.

QUESTION ONE (20 MARKS)

a. Define the following

   i. Open and closed set (3 mark)

   ii. Neighbourhood (2 mark)

b. Suppose that (X_i, d_i) are metric spaces. Show that $X = X_1 \times X_2 \times \ldots \times X_n$ becomes a metric space with the metric d defined by $d(x, y) = \sum_{i=1}^{n} d_i (x_i, y_i)$ (5 marks)

c. Show that A sequence in a metric space (X, d) has at most one limit (6marks)

d. Let U be a subset of the metric space (X, d) then $x \in \overline{U}$ if and only if there exists a sequence (x_n) such that $x_n \to x$ as n $\to \infty$. (4marks)

QUESTION TWO (20 MARKS)

a. Define the following

   i. Open cover (2 mark)

   ii. Relatively compact set (2 mark)

   iii. Continuous function (2 marks)

b. Show that closed subsets of compact metric spaces are compact. (4marks)

c. Show that for every nonempty set $A \subseteq U$ the map $XX \to \mathbb{R}, x \to$dist(x, A) is continuous (5marks)

d. If $f \in C(X, Y)$and X is compact then the image $f(X)$ is compact in Y (5marks)

QUESTION THREE (20 MARKS)

a. Define the following

   i. Normed space (3 mark)

   ii. Banach space (2marks)

   iii. Absolute convergence (2marks)

b. Let $(X, ||.||)$ be a normed space. Show that $||x - y|| \geq |||x|| - ||y|||$ for all $x, y \in X$ (5marks)

c. Show that if a normed space X is complete then every absolutely convergent series in X converges. (7marks)

QUESTION FOUR (20 MARKS)

a. Define the following

   i. Interior point (1marks)

   ii. Metric Space (4marks)

b. Let $p, q \in (1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Show that $ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q$ for all $a, b \geq 0$

(7marks)

c. Let $1 \leq p \leq \infty$ and q the exponent dual to p. Show that $\sum_{i=1}^{N} |x_i||y_i| \leq |x|_p |y|_q$

(8marks)

QUESTION FIVE (20 MARKS)

a. Define the following

   i. Bounded linear operator (2marks)

   ii. Open and Closed Ball (2 marks)

b. Let V and W be normed spaces. Show that if V is finite dimensional then all linear transformations from V to W are bounded. (7marks)

c. Let V, W be normed vector spaces and let T: V ⟶ W be a linear transformation. Show that if T is a bounded linear transformation then T is continuous everywhere in V. (3marks)

a. Let V, W be normed vector spaces and let T: V ⟶ W be a linear transformation. Show that if T is continuous at 0 then T is a bounded linear transformation. (6marks)