(*Knowledge for Development*)

KIBABII UNIVERSITY

UNIVERSITY EXAMINATIONS

2020/2021 ACADEMIC YEAR

FOURTH YEAR FIRST SEMESTER

MAIN EXAMINATION

FOR THE DEGREE OF BACHELOR OF SCIENCE

COURSE CODE: **MAT 403** (MAT 405)

COURSE TITLE: **MEASURE THEORY**

DATE: 20/7/2021 **TIME:** 9 AM - 11 AM

INSTRUCTIONS TO CANDIDATES

Answer Question One and Any other TWO Questions

TIME: 2 Hours

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Question 1 (30 marks) – Compulsory

a) Let A be any bounded subset of real numbers. Define the following:
   i) The outer measure of A (2 mks)
   ii) The inner measure of A (2 mks)

b) For every set A, show that the inner measure is always less than the outer measure. (5 mks)

c) If the function f is a bounded and Lebesque integrable function on the interval $[a,b]$ such that $f(x) = g(x)\ a.e$ on the interval $[a,b]$ then g is Lebesque integrable and $\int_a^b f(x)dx = \int_a^b g(x)dx$ (10 mks)

d) Show that every continuous function is measurable. (3 mks)

e) $f(x) = \begin{cases} 1 & x \text{ is rational} \\ 0 & x \text{ is irrational} \end{cases}$

   Show that this function is Lebesque integrable but is not Riemann integrable. (7 mks)

Question 2 (20 marks)

a) Every bounded measurable function in the interval $[a,b]$ is Lebesque integrable on that interval. Prove. (10 mks)

b) A necessary and sufficient condition for a bounded function f to be Lebesque integrable over the interval $[a,b]$ is that for each given $\varepsilon > 0$, there exists a measurable partition P of the interval $[a,b]$ such that $U(P,f) - L(P,f) < \varepsilon$. Prove. (10 mks)

Question 3 (20 mks)

a) Verify bounded convergence theorem for the sequence of functions
   i) $f_{n}(x) = \frac{1}{(1+\frac{ax}{n})n}$ $0 \leq x \leq 1$ (8 mks)

   ii) $f_n(x) = \frac{1}{(a+\frac{x}{n})n}$ $0 \leq x \leq 1$ (7 mks)

b) $f(x) = \begin{cases} \frac{1}{x^5} & 0 < x < 1 \\ 0 & x = 0 \end{cases}$

   Show that the function f is Lebesque integrable on the interval $[0,1]$ and find the integral. (5 mks)

Question 4 (20 marks)

a) If the functions f_1 and f_2 are measurable on the closed interval [a,b], so are $f_1 + f_2$, $f_1 - f_2$, $f_1 f_2$, $\frac{f_1}{f_2}$; $f_2 \neq 0$ (16 mks)

b) Show that constant functions are measurable. (4 mks)

Question 5 (20 marks)

a) Define the term Lebesque integral. (3 mks)

b) Let f be a bounded function on the interval [a,b], then for any two measurable partitions of the interval $[a, b]$, we have $U(P_{1,}f) \geq L(P_{2,}f)$; $L\int_{-a}^{b} f dx < L\int_{a}^{-b} f dx$. Prove (6 mks)

c) Show that every bounded Riemann integrable function over the interval $[a, b]$ is Lebesque integrable and the two integrals are the same. (6 mks)

d) If the function $f = g\ a.e$ and f is measurable then g is measurable. (5 mks)