(Knowledge for Development)

KIBABII UNIVERSITY

UNIVERSITY EXAMINATIONS

2020/2021 ACADEMIC YEAR

FOURTH YEAR FIRST SEMESTER

MAIN EXAMINATION

FOR THE DEGREE OF BACHELOR OF SCIENCE

COURSE CODE: MAT 407

COURSE TITLE: FUNCTIONAL ANALYSI

DATE: 12/7/2021 **TIME:** 9 AM - 11 AM

INSTRUCTIONS TO CANDIDATES

Answer Question One and Any other TWO Questions

TIME: 2 Hours

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UNIVERSITY EXAMINATIONS, 2020/2021 ACADEMIC YEAR

MAIN EXAMINATION, FOR THE DEGREE OF BACHELOR OF SCIENCE

JULY 2021 COURSE CODE: MAT 407 COURSE TITLE: FUNCTIONAL ANALYSIS

INSTRUCTIONS

Answer QUESTION ONE and any other TWO questions.

QUESTION ONE (30marks)

a). Define the following terms

i). Relatively closed set (2 Marks)

ii). An open ball in a metric space (2 Marks)

iii). A contraction from a metric space to another (2 Marks)

iv). An inner product space (4 Marks)

b). Let V be an inner product space. Prove that $|\langle x, y\rangle| \leq \|x\|\|y\|$ for all $x, y \in V$. (6 Marks)

c). Suppose that X and Y are metric spaces where X is compact and $f: X \to Y$ is continuous. Show that $f(X)$ is bounded. (5 Marks)

d). Prove that the norm $\|x\| = \max\{|x_i|: i = 1, \dots, n\}$ is positively homogeneous. (2 Marks)

e). Prove that the set of piecewise linear functions on $[0,1]$ is dense in the set of continuous functions on a closed interval with respect to the supremum norm, $(C[0,1], \|\cdot\|_\infty)$. (7 Marks)

QUESTION TWO (20marks)

a). Define a compact set, hence show that $\mathbb{R}$ is not compact. (3 Marks)

b). Prove that the set $\left\{\frac{1}{n}: n \in \mathbb{N}\right\} \cup \{0\}$ is sequentially compact. (4 Marks)

c). Let (X, d) be a complete metric space and U_n an open dense subsets of X for $n \in \mathbb{N}$. Prove that $\bigcap_n U_n$ is dense in X. (8 Marks)

d). Define a totally bounded set hence prove that $A = (-1,1)$ is totally bounded. (5 Marks)

QUESTION THREE (20marks)

Consider the function $f(x) = x^2$ defined on $[0, a]$ where $a \in \mathbb{R}$.

a). Find the range of $f(x)$. Is it open or closed, explain? (4 Marks)

b). State and prove the Heine Borel theorem for $\mathbb{R}$. (14 Marks)

c). Use the theorem above to determine if the range of $f(x)$ is compact or not. (2 Marks)

QUESTION FOUR (20marks)

a).(i). Define a nowhere dense set (2 Marks)

(ii). Hence prove that $\mathbb{Z}$ is nowhere dense in $\mathbb{R}$ while $\mathbb{Q}$ is dense in $\mathbb{R}$. (6 Marks)

b). Prove that the intersection of compact sets is compact (5 Marks)

c). Let $B(X)$ be the set of bounded functions on a set X. Prove that the space $(B(X), \|\cdot\|_\infty)$ is complete. (7 Marks)

QUESTION FIVE (20marks)

a). Prove that any continuous function from a compact metric space to any other metric space is uniformly continuous. (7 Marks)

b). (i). When do we say a collection of sets have a finite intersection property. (2 Marks)

(ii). Suppose X is a metric space and $\mathcal{C}$ is a non-empty collection of compact subsets of X. If $\mathcal{C}$ has finite intersection property, then $\bigcap \mathcal{C} \neq \phi$. (3 Marks)

c). State and prove the Nested interval theorem. (8 Marks)