(Knowledge for Development)

KIBABII UNIVERSITY

UNIVERSITY EXAMINATIONS

2020/2021 ACADEMIC YEAR

FIRST YEAR SECOND SEMESTER

SUPPLEMENTARY EXAMINATION

FOR THE DEGREE OF MASTER OF SCIENCE IN PURE MATHEMATICS

COURSE CODE: MAT 811

COURSE TITLE: ABSTRACT INTEGRATION I

DATE: 19/7/2021 **TIME:** 9 AM - 12 NOONJ

INSTRUCTIONS TO CANDIDATES

Answer Any other THREE Questions

TIME: 3 Hours

QUESTION ONE (20 MARKS)

a) Show that given $f: X \rightarrow Y$ and $\mathcal{T}$ is a σ-ring of subsets of Y, then the class of all sets of the form $f^{-1}(B)$ where Bis in $\mathcal{T}$is a σ-ring of subsets of X (5marks)

b)

i. Show that given R is a σ-ring and μ is an extended real valued set function on R which is positive, countably additive ad satisfies the condition $\mu(\phi) = 0$, then μ is a measure (5marks)

ii. Show that given μ is a σ-finite measure on a ring R, then there exists a unique measure $\bar{\mu}$ on $G(R)$ which extends μ. Moreover, $\bar{\mu}$ is σ-finite and $\bar{\mu}(E) = \mu^*(E)$ for all E in $G(R)$ (5marks)

c) Show that given $\alpha_i \uparrow \alpha$ and $\beta_i \uparrow \beta$ then

i. $\alpha_i + \beta_i \uparrow \alpha + \beta$ (3marks)

ii. $\alpha_i \beta_i \uparrow \alpha\beta$ (2marks)

QUESTION TWO (20 MARKS)

a) Define the following terms (4marks)

i. Measurable space

ii. Locally measurable subset of space X

iii. Measurable function

iv. Characteristic function

v. Bored measurable

b)

i. Show that if f is a measurable function and $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ is a Bored measurable function such that $\varphi(0) = 0$, then the composite function $\varphi o f$ is also measurable (4marks)

ii. Show that given f_n is a sequence of measurable functions such that $f_n(x)$ is a bounded sequence, for each x then the functions lim inf f_n and lim sup f_n are also measurable (4marks) c)

i. Show that if f is a measurable function, c is a real valued number and $c > 0$, then $f \cap c$ is a measurable function (4marks)

ii. Show that if f is a measurable function, there exists a sequence of simple functions f_n such that f_n converges to f pointwise on X, that is $f_n(x) \to f(x)$ for each x in X. If moreover $f \geq 0$, show that one can make $0 \leq f_n \uparrow f$ (4marks)

QUESTION THREE (20 MARKS)

a) Define the following terms (5marks)

i. Measure space
ii. Limit superior of a sequence of sets
iii. Limit inferior of a sequence of sets
iv. Contraction
v. Almost everywhere concept

b)

i. Given that $f_n \to f$ a.e show that f_n is fundamental a.e (3marks)

ii. Given that $f_n \to f$ a.e and $f_n \to g$ a.e, then $f = g$ a.e (3marks)

iii. Given that $f_n \to f$ a.e and g is a ral valued function such that $f = g$ a.e then $f_n \to g$ a,e (3marks)

c)

i. State the Egoroff's theorem (3marks)

ii. State the Riesz-weyl theorem (3marks)

QUESTION FOUR (20 MARKS)

a) Define the following terms (4marks)

i. Simple function
ii. Integrable function
iii. Indefinite integral
iv. Mean convergence

b) Show that given f_n is a sequence of integrable functions such that $f_n \leq f_{n+1}$ a.e (n=1,2,…) and $\int f_n \, du$ is bounded, then there exists an integrable function f such that $f_n \uparrow f$ a.e . Necessarily, $\int f_n \, du \uparrow \int f \, du$. Indeed, if h is any measurable function such that $f_n \to h$ a.e then $h = f$ a.e, h is integrable and $\int f_n \, du \uparrow \int h \, du$ (10marks)

c) Given f_n, f, g_n, g are integrable functions, c is a real number and A is a locally measurable set. Assume that $f_n \to f$ in mean and $g_n \to g$ in mean. Show that (6marks)

i. $cf_n \to cf$ in mean

ii. $f_n + g_n \to f + g$ in mean

iii. $|f_n| \to |f|$ in mean

iv. $f_n \cup g_n \to f \cup g$ in mean and $f_n \cap g_n \to f \cap g$ in mean

v. ${f_n}^+ \to f^+$ in mean and ${f_n}^- \to f^-$ in mean

vi. $\chi_A f_n \to \chi_A f$ in mean

QUESTION FIVE (20 MARKS)

a) Show that if f_n is a sequence of integrable functions such that $f_n \geq 0$ a.e and lim inf $\int f_n \, du < \infty$ then there exists an integrable function f such that $f = lim\, inf\, f_n$ a.e and one has $\int f_n \, du \leq$ lim inf $\int f_n \, du$ (6marks)

b)

i. Show that if $f, g \in \mathcal{L}^2$ then $(f|g) \leq \| f \|_2 \| g \|_2$ (3marks)

ii. Show that if $f, g \in \mathcal{L}^2$ then $\| f + g \|_2 \leq \| f \|_2 + \| g \|_2$ (3marks)

iii. Given that $f_n \to f$ a.e show that f_n is fundamental a.e (3marks)

iv. Given that $f_n \to f$ a.e and $f_n \to g$ a.e, then $f = g$ a.e (3marks)

c) Given that f_n is a 2-mean fundamental sequence in $\mathcal{L}^2$ show that there exists a function f in $\mathcal{L}^2$ such that $f_n \to f$ in 2-mean (8marks)