(*Knowledge for Development*)

KIBABII UNIVERSITY

UNIVERSITY EXAMINATIONS

2020/2021 ACADEMIC YEAR

FOURTH YEAR FIRST SEMESTER

MAIN EXAMINATION

FOR THE DEGREE OF BACHELOR OF SCIENCE

COURSE CODE: STA 443

COURSE TITLE: PROBABILITY AND MEASURE

DATE: 15/7/2021 **TIME:** 2 PM - 4 PM

INSTRUCTIONS TO CANDIDATES

Answer Question One and Any other TWO Questions

TIME: 2 Hours

QUESTION ONE (30 MARKS)

1. (a) Define the following terms (1 mk)
   i. Probability space (1 mk)
   ii. Probability measure (1 mk)
   iii. Sigma-algebra (1 mk)
   iv. Measurable space

   (b) Suppose that $A, B \in \mathcal{A}$. Show that $\mu(B) = \mu(A \cap B) + \mu(B \cap A')$ (3 mks)

   (c) Let $\{F_i \subset R^n : i \in N\}$ is countable collection of R^n. Show that

$$\sum_{i=1}^{\infty} \mu^*(F_i) \geq \mu^*(\cup_i^{\infty} F_i)$$

(4 mks)

   (d) State two properties of probability measure (2 mks)

   (e) Let $0 \leq f_n \to f$ almost everywhere and $\int f_n d\mu \leq A < \infty$, show that f is integrable and $\int f d\mu \leq A$ (5 mks)

   (f) Let X and Y be independent random variables. Show that

$$E[X|Y = y] = E[X]$$

(4 mks)

   (g) Find the integral $f(x, y) = x^2 + y^2$, on the domain

$$D = \left\{(x, y) \in R^2 : 0 < x < 2, x^2 < y < x\right\}$$

(3 mks)

   (h) Suppose (X, δ, μ) is a measure space and f and g are measurable functions on X and $A, B \in \delta$. State three properties of f and g. (3 mks)

   (i) Prove that if $\mu^*(A) = 0$ then for each B, $\mu^*(A \cup B) = \mu^*(B)$ (2 mks)

QUESTION TWO (20 MARKS)

2. (a) Let μ be a δ-finite measure on an algebra $\mathcal{A}$ of subsets of ω. Show that:

   i. there exists an increasing sequence (5 mks)

   ii. there exists a disjoint δ-finite sequence (5 mks)

   (b) Suppose $\{B_n\}$ is sequence of independent events and $\sum_n P\{B_n\} = \infty$. Show the probability that B_n occurs infinitely often is one. (10 mks)

QUESTION THREE (20 MARKS)

3. (a) Let f_1 and f_2 be measurable functions on a common domain. Show that each set $\{\omega : f_1(\omega) < f_2(\omega)\}$, $\{\omega : f_1(\omega) = f_2(\omega)\}$ and $\{\omega : f_1(\omega) > f_2(\omega)\}$ is measurable (8 mks)

   (b) Suppose $f = \sum_i x_i I_{Ai}$ is a non negative simple function, $\{A_i\}$ being decomposition of S into F sets, show that

$$\int f d\mu = \sum_i x_i \mu(A_i)$$

(6 mks)

   (c) Let $P, q, r \in [1, \infty]$ satisfy $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Prove that for all measurable f and g defined on a space $(X, \mathcal{A}, \mu)$, we haven $||fg||_r \leq ||f||_p ||g||_q$ (6 mks)

QUESTION FOUR (20 MARKS)

4. (a) What are Lebesgue measurable sets? (2 mks)

   (b) Describe any two Lebesgue measurable sets (4 mks)

   (c) State and explain any four measurable functions (8 mks)

   (d) Show that if $\{f_n\}$ is a sequence of non-negative measurable functions, and $\{f_n(x) : n \leq 1\}$ increases monotonically to $f(x)$ for each x then

$$\lim_{n \to \infty} \int_E f_n(x) dm = \int_E f dm$$

(6 mks)

QUESTION FIVE (20 MARKS)

5. (a) State and explain two properties of conditional expectation (4 mks)

   (b) Find the mathematical expectation of a random variable with:

      i. uniform distribution over the interval $[a, b]$

      ii. triangle distribution

      iii. exponential distribution (6 mks)

   (c) Let $f_n \geq 0$ be a measurable function. Show that $\int_x \liminf f_n d\mu \leq \liminf \int_x f_n d\mu$ as $n \to \infty$ (10 mks)