(Knowledge for Development)

KIBABII UNIVERSITY

UNIVERSITY EXAMINATIONS

2019/2020 ACADEMIC YEAR

FIRST YEAR FIRST SEMESTER

MAIN EXAMINATION

FOR THE DEGREE OF MASTER OF SCIENCE IN PURE MATHEMATICS

COURSE CODE: **MAT 812**

COURSE TITLE: **GROUP THEORY**

DATE: 17/02/2021 **TIME: 2 PM -5 PM**

INSTRUCTIONS TO CANDIDATES

Answer Any THREE Questions

TIME: 3 Hours

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KIBABII UNIVERSITY

MATHEMATICS DEPARTMENT

FIRST SEMESTER EXAMS, JANUARY 2020

MAT 812: GROUP THEORY I

Answer Any Three Questions

Time: 3 hours

QUESTION ONE (20 MARKS)

a. Suppose that a group G acts on a set X. Let B(x) be the orbit of $x \in X$, and let stab(x) be the stabilizer of x. Show that the size of the orbit is the index of the stabilizer i.e. $|B(x)| = [G: Stab(x)]$. If G is finite, then $|B(x)| = |G|/|Stab(x)|$. (10 marks)

b. Let the finite group G act on the finite set X, and denote by X_g the set of elements of X that are fixed by g, that is $X_g = \{x \in X, g.x = x\}$. Show that the number of orbits = $1/|G|\sum_{g\in G} |X^g|$, that is the number of orbits is the average number of points left fixed by elements of G. (6 marks)

c. Define the following

   i. Transitive action (2 marks)

   ii. The stabilizer of an element (2marks)

QUESTION TWO (20 MARKS)

a. Show that every finite group G has a composition series (5 marks)

b. Show that if H is a normal subgroup of a finite group G and if H and G/H are both soluble then G is soluble. (5 marks)

c. Show that all finite Abelian groups are soluble (6marks)

d. Define the following

   i. Composition series (3 marks)

   ii. Soluble group (1 marks)

QUESTION THREE (20 MARKS)

a. Show that every nilpotent group is solvable. (3 marks)

b. Show that a group G is nilpotent iff it has a central series. (3marks)

c. Show that if G is a finite group and P is a Sylow p-subgroup of G then $N_G(N_G(P)) = N_G(P)$ (6marks)

d. Show that if H is a proper subgroup of a nilpotent group G, then H is a proper subgroup of $N_G(H)$. (8marks)

QUESTION FOUR (20 MARKS)

a. Show that if G is the internal direct product of H and K, then G is isomorphic to the external direct product $H \times K$. (8marks)

b. Let H and K be groups, and let $\rho : K \rightarrow Aut(H)$ be a group homomorphism. Show that the binary operation $(H \times K)\times(H \times K) \rightarrow (H \times K)$, endows $H \times K$ with a group structure, with identity element (1,1). (6 marks)

c. Suppose that G is a group with subgroups H and K, and G is the internal semi-direct product of H and K. Show that $G \simeq H \times_\rho K$ where $\rho: K \rightarrow Aut(H)$ is given by $_{\rho k}(h) = khk^{-1}$, $k \in K$, $h \in H$. (6marks)

QUESTION FIVE(20 MARKS)

a. Show that any cyclic abelian group is isomorphic to Z or Z_n, for some n. (5 marks)

b. If k = mn, where m and n are relatively prime integers, then Z_k is isomorphic to $Z_m \oplus Z_n$. (5 marks)

c. Show that if X is a basis for a free group F then X generates F. (5marks)

d. Suppose F is a free group with basis X and G is a free group with basis Y. Then $F \cong G$ if $|X| = |Y|$ (5marks)