(Knowledge for Development)

KIBABII UNIVERSITY

UNIVERSITY EXAMINATIONS

2019/2020 ACADEMIC YEAR

FOURTH YEAR SECOND SEMESTER

MAIN EXAMINATION

FOR THE DEGREE OF BACHELOR OF SCIENCE

(MATHEMATICS)

COURSE CODE: **MAT 424**

COURSE TITLE: ORDINARY DEFFERENTIAL EQUATION III

DATE: 13/11/2020 **TIME:** 9.00 AM- 11.00 AM

INSTRUCTIONS TO CANDIDATES

Answer Question One and Any other TWO Questions

TIME: 2 Hours

This Paper Consists of 4 Printed Pages. Please Turn Over.

QUESTION ONE (30 MARKS)

a) Define the following terms (4 Marks)

(i) Stability

(ii) Equilibrium solution

b) Discuss the existence and unique solution for the IVP

$y' = \frac{2y}{x}, y(x_o) = y_o$ (5 Marks)

c) Consider the ODE $y' = xy - \sin y, y(0) = 2$, Show that there exists unique solution in the neighbourhood of (0,2) (5Marks)

d) Show that the solution to the differential equation of RL circuit $RI + L\frac{dI}{dt} = V$ is given by

$I = \frac{V}{R}\left(1 - e^{-\left(\frac{R}{L}\right)t}\right)$ (6 Marks)

e) Prove that every fundamental matrix solution $X(t)$ of $\dot{x} = Ax$ has the form where

$$X(t) = P(t)e^{Bt}$$

Where $P(t) = P(t+T)$ for all $t \in \mathbb{R}$, is a non-singular matrix and B is also an $n \times n$ constant matrix. (5 Marks)

f) Solve the initial value problem $\dot{x} = \beta x \quad x(0) = x_0$ using Picards method of successive approximation. (5 Marks)

QUESTION TWO (20 MARKS)

a) Find the solution to the IVP: $X' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} X, \ x(0) = x_O$ (5Marks)

b) Show that the origin is unstable focus for this system and use the Poincare Bendixson Theorem to show that there is periodic orbit in the annular region $D_2 = \{x \in \mathbb{R}^2 \, | 1 < |x| < 2\}$ (4 Marks)

c) Solve the following IVP $y''' - 5y'' - 22y' + 55y = 0$ where $y(0) = 1$, $y'(0) = -2$, $y''(0) = -4$ (6 Marks)

d) Proof that if f and $\frac{\partial f}{\partial y}$ are continuous on R^2 and ϕ is a solution of $y' = f(t, y)$ and $y(0) = 0$ on some interval I containing O, then ϕ is the unique solution on this interval (5Marks)

QUESTION THREE (20 MARKS)

a) Prove that if $\phi(t)$ is a fundamental matrix for the system $x' = A(t)x$, if its determinant $|\phi(t)|$ is non-zero and it satisfies the matrix equation $\phi' = A\phi$ where ϕ' means that each entry ϕ has been differentiated. (6 Marks)

b) Find the general solution of the non-homogeneous equation $y''' + 3y'' - 10y' = x - 3$ (8 Marks)

c) Linearize the system at each of the equilibrium points and determine the behaviour of the solutions near the equilibrium points (6Marks)

QUESTION FOUR (20 MARKS)

a) Define the following terms (4 Marks)
   (i) Liapunov function
   (ii) Limit cycle

b) Prove that the function $V(y_1, y_2) = y_1^2 + y_1^2 y_2^2 + y_2^4 \qquad (y_1, y_2) \in \square^2$

Is a strict Liapunov function for the system

$$\dot{x}_1 = 1 - 3x_1 + 3x_1^2 + 2x_2^2 - x_1^3 - 2x_1x_2^2$$

$$\dot{x}_2 = x_2 - 2x_1x_2 + x_1^2x_2 - x_2^3$$

At fixed point (1,0) (6 Marks)

c) Show that the phase portrait of

$$\ddot{x} - (1 - 3x^2 - 2\dot{x})\dot{x} + x = 0$$

Has a limit cycle (5 Marks)

d) Find the derivative of the function

$$f(x) = \begin{pmatrix} x_1 - x_2^2 \\ -x_2 + x_1x_2 \end{pmatrix} = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$$

And evaluate it at the point $x_0 = (1, -1)^T$ (5Marks)

QUESTION FIVE (20 MARKS)

Consider the differential equations that model the populations $x_1(t)$ and $x_2(t)$ at time $t \geq 0$ of two competing species

$$\dot{x}_1 = ax_1(1 - x_1) - bx_1x_2$$
$$\dot{x}_2 = cx_2(1 - x_2) - dx_1x_2$$

Let $a = 1, b = 2, c = 1$ and $d = 3$

(i) On one phase plane sketch the isoclines of the differential equations (5) and determine all its equilibriums (4 Marks)

(ii) Determine the type of stability of all equilibrium points in (i) above (6 Marks)

(iii) Sketch the phase plane and clearly indicate the direction of the vector field defined by the equations above. (4 marks)

(iv) State algebraically and sketch by shading appropriately the basin of attraction of each attracting fixed point. (4 Marks)

(v) If a=3, b=2, c=4 and d=3. Show that the populations co-exist at some point $\bar{x}\left(\frac{2}{3}, \frac{1}{2}\right)$ (2 Marks)