(Knowledge for Development)

KIBABII UNIVERSITY

UNIVERSITY EXAMINATIONS

2019/2020 ACADEMIC YEAR

FIRST YEAR FIRST SEMESTER

MAIN EXAMINATION

FOR THE DEGREE OF MASTER OF SCIENCE IN PURE MATHEMATICS

COURSE CODE: MAT 824

COURSE TITLE: OPERATOR THEORY I

DATE: 17/02/2021 **TIME:** 8 AM -11 AM

INSTRUCTIONS TO CANDIDATES

Answer Any THREE Questions

TIME: 3 Hours

MAT 824: OPERATOR THEORY I

Answer Any Three Questions

Time: 3 hours

QUESTION ONE (20 MARKS)

a. Show that if X is an inner product space then $||x||^2 = < x, x >^{1/2}$ defines a norm on X. (5 marks)

b. Show that in an inner product space, inner product is continuous (5 marks)

c. Show that every inner product space is uniformly convex (5 marks)

d. Show that if $T \in B(H)$ then there is a unique $U \in B(H)$ such that $< Tx , y > = < x, Uy>$ for all $x, y \in H$ (5 marks)

QUESTION TWO (20 MARKS)

a. Show that if $T \in B(H)$ and $< Tx , x > = 0$ then $T = 0$ for all $x \in H$ (5 marks)

b. Show that the Hilbert adjoint T^* is a bounded linear operator with $|| T^*|| = || T||$ (5 marks)

c. Show that if M is a closed subspace of H then pm is a projection having range M. (5 marks)

d. Show that if M is a closed subspace of Hilbert space H and $x \in H$ then there is a unique element $y \in M$ and $z \in M^{\perp}$ such that $x = y + z$ (5 marks)

QUESTION THREE (20 MARKS)

For $T, T_1, T_2 \in B(H)$ and $\alpha \in C$, show that

a. $(T_1 + T_2) ^* = T_1^* + T_2^*$ (4marks)

b. $(AT)^* = \alpha^- T^*$ (3marks)

c. $(T_1T_2)^* = T^*_2T^*_1$ (3marks)

d. $||T^*T|| =|| T ||^2$ (10 marks)

QUESTION FOUR (20 MARKS)

a. Let $T \in B(H,K)$. Show that there exists at most one operator $S \in B(K,H)$ such that $ST = I$ and $TS = I$ (3 Marks)

b. Let $S \in B(H,K)^{\times}$ and $T \in B(K,L)^{\times}$. Show that the operator $TS \in B(H,L)^{\times}$, with $(TS)^{-1} = S^{-1}T^{-1}$ (4Marks)

c. If $T \in B(H)$ is such that $||T|| < 1$, show that $I - T$ is invertible and the series $\sum_{n=0}^{\infty} T^n = 1 + T + T^2 + \ldots\ldots\ldots\ldots$ is absolutely convergent, its sum equals $(I - T)^{-1}$ and $||(I - T)^{-1}|| \leq (1 - ||T||)^{-1}$ (8 Marks)

d. If $T \in B(H)$ is self adjoint i.e., $T = T^*$, show that $\rho(T) = ||T||$. (5 marks)

QUESTION FIVE (20 MARKS)

a. If $T \in B(H)^{\times}$ is invertible, show that $T^{-1} \in B(H)^{\times}$ (3Marks)

b. Let $T \in B(H)^{\times}$ be invertible. Show that the square root $T^{\frac{1}{2}}$ is invertible and $\left(T^{-1}\right)^{\frac{1}{2}} = \left(T^{\frac{1}{2}}\right)^{-1}$. (5 Marks)

c. Let $T \in B(H)^{\times}$. Show that T is invertible if and only if T is uniformly positive (10 marks)

d. If $T \in B(H;K)$ is invertible show that $|T| \in B(H)^{\times}$ (2marks)