(*Knowledge for Development*)

KIBABII UNIVERSITY

UNIVERSITY EXAMINATIONS

2020/2021 ACADEMIC YEAR

MAIN CAMPUS

MASTERS FIRST YEAR SECOND SEMESTER EXAMINATIONS

COURSE CODE: STA 806

COURSE TITLE: THEORY OF LINEAR MODELS

DATE: 18/5/2021 **TIME:** 9:00 A.M – 12:00 NOON

INSTRUCTIONS TO CANDIDATES:

Answer Question one and any other two questions.

QUESTION ONE (30 MARKS)

a) Consider a linear regression model.

Show that the model can be written in matrix form as

$\underline{Y} = X\beta + \underline{\varepsilon}$, Where $\underline{Y}$, β and $\underline{\varepsilon}$ are vectors of order n×1; (k+1) ×1 and n×1 respectively, while X is a matrix of order n×(k+1). (4marks)

$\underline{\hat{\beta}} = (X^TX)^{-1}X^TY$ (4marks)

b) Let $S^2 = \frac{1}{n-k-1}\sum_{i-1}^{n}(y_i - \underline{X}_i^T\underline{\hat{\beta}})^2$ where $\underline{X}_i^T$ is the i-th row of the matrix X.

Show that if $Var(\underline{\varepsilon}) = \sigma^2 I$ then $E(S^2) = \sigma^2$ (6marks)

c) In part (b) Let $\hat{Y} = \hat{\beta}_0 + \hat{\beta}_1 X$ be the predicted value of Y. Let $\underline{X}^T = (1 \;\; X)$ such that $\hat{y} = \hat{\beta}_x^T \underline{X} + \varepsilon$

Show that if $\underline{Z}^T = (1 \;\; cx)$ where c $\hat{y} = \hat{\beta}_x^T \underline{X}$ where $\hat{\beta}_z$ is the Least square estimator of $\underline{\beta}$ assuming $Y = \underline{\beta}^T \underline{Z} + \varepsilon$ (6marks)

d) Let $\underline{Y} = \beta_0 + \beta_1 X + \varepsilon$ where ε is the error term. Using results in part (b) or, otherwise deuce the least square estimators of β_o and β_1 say $\hat{\beta}_0$ and $\hat{\beta}_1$ respectively

Show that

(i). $E(\hat{\beta}_0) = \beta_o$ (ii). $E(\hat{\beta}_1) = \beta_1$

Determine

(Iii). Var ($\hat{\beta}_0$) (iv). Var ($\hat{\beta}_1$)

(v). Cor ($\hat{\beta}_0$, $\hat{\beta}_1$) (10 marks)

QUESTION TWO (20 MARKS)

e) Show that if E(Y) =X$\underline{\beta}$ and Cor ($\underline{Y}$) = $\sigma^2 I$ then the least square estimators $\hat{\beta}_j$, j = 0,1,…,k, have minimum variance among all linear unbiased estimators. (10marks)

b) Using part (a) or otherwise, show that if E($\underline{Y}$) = X$\underline{\beta}$ and Cov($\underline{Y}$) = $\sigma^2 I$, then the best linear unbiased estimator of $\underline{a}^T\underline{\beta}$ is $\underline{a}^T\hat{\underline{\beta}}$ where $\hat{\underline{\beta}}$ is the least square estimator of $\underline{\beta}$. (5marks)

f) Does the results in part (a) rely on the distribution of the random vector $\underline{Y}$? Comment (5marks)

QUESTION THREE (20 MARKS)

a) Show that if $\underline{Y} \sim N_n[X\underline{\beta}, \sigma^2 I]$ where X is n×(k+1) matrix of rank k+1<n, the maximum likelihood estimator of $\underline{\beta}$ and σ^2 are $\hat{\underline{\beta}} = (X^TX)^{-1}X^T\underline{Y}$

and $\hat{\sigma}^2 = \frac{1}{n}(\underline{Y} - X\hat{\underline{\beta}})^T(Y - X\hat{\underline{\beta}})$ (10 marks)

b) Using the results in part (a), or otherwise, Show that

(i). $\hat{\underline{\beta}}$ is $N_{k+1}[\underline{\beta}, \sigma^2(X^TX)^{-1}]$

(ii) $n\frac{\hat{\sigma}^2}{\sigma^2}$ is chi-square with degrees of freedom = n-k-1

(iii). $\hat{\underline{\beta}}$ and $\hat{\sigma}^2$ are independent. (10 marks)

QUESTION FOUR (20 MARKS)

Consider the data in the following table

Observation Number	Y	X_1	X_2
1	2	0	2
2	3	2	6
3	2	2	7
4	7	2	5
5	6	4	9
6	8	4	8
7	10	4	7
8	7	6	10
9	8	6	11
10	12	6	9
11	11	8	15
12	14	8	13

a) Show how the date can be modeled by a regression model given by

$\underline{Y} = X\underline{\beta} + \underline{\varepsilon}$ (5marks)

(b). Compute the least square estimate of $\hat{\underline{\beta}}$. (5marks)

(c). If Var($\underline{\varepsilon}$) = $\sigma^2 I$ is known, calculate Var ($\hat{\underline{\beta}}$). (5marks)

(d). Calculate the estimate of the estimator S^2, defined in question one (b).

(5 marks)

QUESTION FIVE (20 MARKS)

Suppose we fit the model $\underline{Y} = X_1\beta_1^x + \varepsilon^x$ when the correct model is

$\underline{Y} = X_1\beta_1 + X_2\beta_2 + \underline{\varepsilon}$ with

$Cov(\underline{Y}) = \sigma^2 I$

(a). Show that (i) the best square estimator for

$$\hat{\underline{\beta}}_1^x = (X^T X)^{-1} X_1^T Y$$

(ii) $E(\hat{\underline{\beta}}_1^x) = \underline{\beta}_1 + \underline{\beta}_2$, where $A = (X_1^T X_1)^{-1} X_1^T X_2$

(iii). $Cov(\hat{\underline{\beta}}_1^x) = \sigma^2 (X_1^T X_1)^{-1}$

(iv). If columns of X_1 are orthogonal to columns of X_2 then $\hat{\underline{\beta}}_1^x$ is unbiased.

(b). Let $\hat{\underline{\beta}} = (X^T X)^{-1} X^T Y$ from the full model be partitioned as $\hat{\underline{\beta}} = \begin{pmatrix} \hat{\underline{\beta}}_1 \\ \hat{\underline{\beta}}_2 \end{pmatrix}$, and let

$\hat{\underline{\beta}}_1^x = (X^T X)^{-1} X_1^T Y$ be the estimator for the reduced model.

Show that

(a). $Cov(\hat{\underline{\beta}}_1) - Cov(\hat{\underline{\beta}}_1^x) = \sigma^2 A B^{-1} A^T$, which is a positive definite matrix, where

$A = (X_1^T X_1)^{-1} X_1^T X_2$ $\quad\quad$ $B = X_2^T X_2 - X_2^T X_1 A$

(b). $Var(\hat{\beta}_j) > Var(\hat{\beta}_j^*)$

(c). $Var(\underline{X}_0^T \hat{\underline{\beta}}) \geq Var(X_{01}^T \hat{\underline{\beta}}_1^x)$

where $\underline{X}_0 = \begin{pmatrix} X_{01} \\ X_{02} \end{pmatrix}$