(Knowledge for Development)

KIBABII UNIVERSITY

UNIVERSITY EXAMINATIONS

2020/2021 ACADEMIC YEAR

FIRST YEAR SECOND SEMESTER EXAMINATIONS

FOR THE DEGREE OF MASTER OF SCIENCE IN STATISTICS

COURSE CODE: STA 813

COURSE TITLE: BAYESIAN INFERENCE

DATE: 17/5/2021 **TIME:** 9:00 A.M – 12:00 NOON

INSTRUCTIONS TO CANDIDATES

- Answer question ONE (COMPULSORY) and any other TWO questions

QUESTION ONE (30 MARKS)

a) Define the following terms

i. Posterior distribution

ii. Prior distribution

iii. Loss function

b) A random sample $X_1, X_2, ..., X_n$ is drawn from a normal population $N(\theta, \sigma^2)$ where σ^2 is known but θ unknown. Obtain

i. The likelihood function

ii. The posterior distribution if a prior $N(\theta_1, \sigma_0^2)$ is used.

iii. The Bayes estimate of θ

iv. Bayes risk

c) Let $P(X \mid \lambda)$ be the Poisson probability density. If the prior $P(\lambda)$ is distributed as gamma with parameters α *and* β both parameters known. Obtain

i. Bayes estimator for λ

ii. Bayes risk

QUESTION TWO (20 MARKS)

a) Let $X_1, X_2, ..., X_n$ be a random sample from the density

$$f(X;\theta) = \begin{cases} \theta^X (1-\theta)^{1-X} & X = 0,1 \quad 0 \le \theta \le 1 \\ 0 & elsewhere \end{cases}$$

Find the Bayes estimator of θ with prior taken as $P(\theta) = 1$.

b) Find the Bayes estimator of θ if

$$f(X;\theta) = \begin{cases} \dfrac{2X}{\theta} & 0 < X < \theta \\ 0 & elsewhere \end{cases}$$

Take the prior $P(\theta) = 1$. Use the loss function $L(\delta(\underline{X}, \theta) = \theta^2 (\delta(\underline{X} - \theta)^2$ to obtain

i. Bayes estimator for θ

ii. Bayes risk

QUESTION THREE (20 MARKS)

a) Define the following terms

   i. Improper prior

   ii. Exponential family

b) Show that the prior of the form $(g(\theta))^b \exp\sum_{i=1}^{n}\eta_i(\theta)a_i$ combines well with the exponential family.

c) If X is distributed as binomial with parameter $n \, and \, \theta$, where n is known. Show that the non-information prior for θ is $Beta(\frac{1}{2},\frac{1}{2})$.

QUESTION FOUR (20 MARKS)

a) Consider a binomial distribution $B(n,\theta)$. Compute the conjugate prior for θ.

b) For an estimation problem

$$X_i \mid \theta \sim u(0,\theta) : i = 1,2,\ldots,n$$

$$\frac{1}{\theta} \mid a,b \sim G(a,b) : a,b \ known$$

Sufficiency allows us to work with the density of $Y = \max(X_1, X_2, \ldots, X_n)$ which is given by $g(y \mid \theta) = \frac{ny^{n-1}}{\theta^n} \quad 0 < y < \theta$

Obtain the Bayes estimator of θ hence Bayes risk.

QUESTION FIVE (20 MARKS)

a) Define the following terms

   i. Bayes estimator

   ii. Bayes risk

   iii. MSE

b) Suppose a random sample of 5 observation is drawn from a uniform distribution on the interval $(0,\theta)$ and that the prior is

$$p(\theta) = \begin{cases} \frac{1}{\theta^2} & \theta > 1 \\ 0 & elsewhere \end{cases}$$

The values of the observations in the sample are: 0.5, 0.7, 0.8, 0.2 and 0.3. Determine the Bayes estimator of θ.

c) Suppose that $X_1, X_2, ..., X_n \sim N(\theta, \sigma^2)$, where both θ and σ^2 are unknown when the prior assigns to $\tau = \frac{\sigma^2}{2}$ the distribution $G(\alpha, \beta)$, and take θ to be independent of τ for the sake of simplicity and assume the uniform improper prior $d\theta$ *for* θ.

Hint: substitute $\frac{\sigma^2}{2}$ *by* τ in the normal distribution and find the estimator of τ instead of σ^2. Obtain Bayes estimator for θ *and* σ^2.