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(Knowledge for Development)

KIBABII UNIVERSITY

UNIVERSITY EXAMINATIONS

2017/2018 ACADEMIC YEAR

FOURTH YEAR FIRST SEMESTER

SPECIAL/SUPPLEMENTARY EXAMINATION

FOR THE DEGREE OF BACHELOR OF SCIENCE

(MATHEMATICS)

COURSE CODE: **STA 443**

COURSE TITLE: **PROBABILITY AND MEASURE**

DATE: 05/10/18 **TIME:** 3 PM -5 PM

INSTRUCTIONS TO CANDIDATES

Answer Question One and Any other TWO Questions

TIME: 2 Hours

ANSWER QUESTION ONE AND ANY OTHER TWO QUESTIONS

QUESTION ONE (30 MARKS)

1. (a) Let $\mathcal{F}$ be a collection of subsets of X

    i. State three conditions that satisfy $\mathcal{F}$ to be a sigma algebra (3 mks)

    ii. Give three examples of σ-algebras (3 mks)

   (b) State the basic axioms that must be satisfied by a probability measure (3 mks)

   (c) If μ is a σ-finite measure on an algebra A of subsets of S. Show that:

    i. there exists an increasing sequence and (4 mks)

    ii. there exists a disjoint σ-finite sequence. (4 mks)

   (d) If $A \subset B$, show that $\mu^*(A) \leq \mu^*(B)$. (3 mks)

   (e) Suppose A and B are independent events in the sample space. Show that A^c and B are independent. (5 mks)

   (f) Prove that every monotone function is measurable. (5 mks)

QUESTION TWO (20 MARKS)

2. (a) Show that a σ-field cannot be countably infiniteits cardinality must be finite or else at least that of the continuum. Show by example that a field can be countably infinite. (5 mks)

   (b) Let $f_{XY}(x, y) = \frac{1}{30}(x^2 + y^2)$ if $0 < x < 2, 1 < y < 4$ and zero otherwise. Find $P(X + Y > 4)$ (5 mks)

   (c) Suppose X is a random variable with distribution μ_X, and g is a Borel measurable function. Show that

$$E[g(X)] = \int_R g(x) d\mu_X$$

(5 mks)

   (d) State and briefly explain any two types of measures on the intervals over the real line. (5 mks)

QUESTION THREE (20 MARKS)

3. (a) Suppose $X_1, X_2, \ldots, X_n$ are random variables with finite variance. If $X_1, \ldots, X_n$ are pairwise orthogonal. Show that

$$Var[X_1 + X_2 + ... + X_n] = Var[X_1] + ... + Var[X_n]$$

(8 mks)

   (b) Suppose $f = \sum_i x_i I_{Ai}$ is a non-negative simple function, $\{A_i\}$ being decomposition of S into F sets, show that

$$\int f d\mu = \sum_i x_i \mu(A_i)$$

(6 mks)

   (c) Let $r, s, t \in [1, \infty]$ satisfy $\frac{1}{r} + \frac{1}{s} = \frac{1}{t}$. Prove that for all measurable f and g defined on a space (X, A, μ), given $||fg||_t \leq ||f||_r ||g||_s$ (6 mks)

QUESTION FOUR (20 MARKS)

4. (a) State and explain two properties of conditional expectation (4 mks)

   (b) Find the mathematical expectation of a random variable with:

      i. uniform distribution over the interval $[a, b]$

      ii. triangle distribution

      iii. exponential distribution (6 mks)

   (c) Show that if $\{f_n\}$ is a sequence of non-negative measurable functions, and $\{f_n(x) : n \leq 1\}$ increases monotonically to $f(x)$ for each x then

$$\lim_{n \to \infty} \int_E f_n(x) dm = \int_E f dm$$

(5 mks)

   (d) Find the integral $f(x, y) = x^2 + y^2$, on the domain

$$D = \left\{(x, y) \in R^2 : 0 < x < 1, x^2 < y < x\right\}$$

(6 mks)

QUESTION FIVE (20 MARKS)

5. A σ-field is countably generated, or separable, if it is generated by some countable class of sets.

   (a) Show that the σ-field $\mathcal{B}$ of Borel sets is countably generated. (6 mks)

   (b) Show that the σ-field is countably generated if and only if Ω is countable. (7 mks)

   (c) Suppose that $\mathcal{F}_1$ and $\mathcal{F}_2$ are σ-fields, $\mathcal{F}_1 \subset \mathcal{F}_2$, and $\mathcal{F}_2$ is countably generated. Show by example that may not be countably generated. (7 mks)